Notes on Quantum Mechanics
Lectures by Prof. Barton Zwiebach
MIT OCW Physics 8.05

Herein find notes from Barton Zwiebach's lectures on Quantum Mechanics, Physics 8.05 in MIT OpenCourseware. The full set of course notes is available on the MIT OCW web site, and they're really complete and helpful. These here are just the main new ideas I've learned and my extrapolations from them. Zwiebach is an extraordinary teacher, and he's clarified a whole bunch of concepts for me.

Lecture 1: the Schrodinger equation

At first, I thought Oh, no. Here's a rerun of the wavefunction. My mind goes numb.

This is different. Zwiebach is so enthusiastic and so clear it actually begins to make sense.

Start with the Schrodinger general equation.

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \left(\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x,t) \right) \Psi(x,t)$$

Note the symbols. Capital Psi refers specifically to the general equation. We'll see later that little psi refers to the time-independent equation

$$\hat{H}\psi(x) = E\psi(x)$$

Those parentheses on the right of the general equation by the way. That's $\hat{H}$, the time operator Hamiltonian. The total energy.

Allowed potentials include, among the more common, square well (and related step functions), parabolic well, and delta function:

Constraints on $V(x)$

bounded

δ

delta function

unbounded well
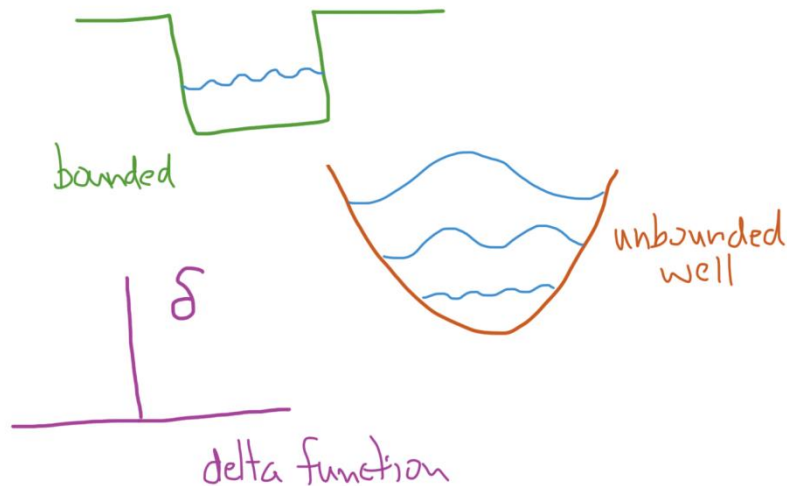
Develop the mathematical tools. I'll drop the function parameters, for clarity. But keep in mind Ψ means $\Psi(x,t)$, while ψ means $\psi(x)$.

If Ψ represents a particle, we need maths to locate the particle in space and time. Ψ is complex. It has to be, given that i on the l.h.s. of the general equation. In order to locate particles in the real world we need real values. Logical choice is the metric from complex math, the density function

$$\rho(x,t) \equiv \Psi^*\Psi$$

Consider position first. Main idea, as usual for unitarity in probability, is that the particle exists, with probability one, so we must be able to find it somewhere. In one dimension, the particle has to be somewhere along the real number line in the range minus infinity to plus infinity. So

$$\int_{-\infty}^{+\infty} \Psi^*\Psi \, dx = 1$$

Note that in the integral, $\Psi^*\Psi \, dx$ is the probability density of finding the particle somewhere in the interval $(x, x + dx)$.

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All that is familiar. Unitary probability. Amplitude vs. probability. New is better understanding of the density function. I can sort of see it now; there it is on the Real line.
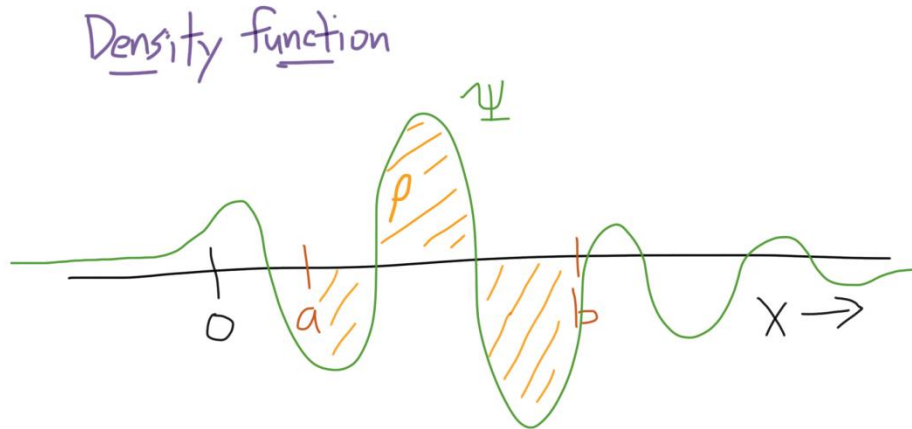

Density function

Figure. Probability density. Absolute value of area, orange, between a and b gives probability of finding the particle in that interval.

There are conditions at infinity. Both Psi and $\frac{\partial \Psi}{\partial x}$ have to go to zero at $\pm \infty$. Otherwise probability and momentum, among other things, blow up.

Next is the continuity equation for the wavefunction. Here's a puzzlement. It's easy to visualize charge conservation, for example, in its continuity equation

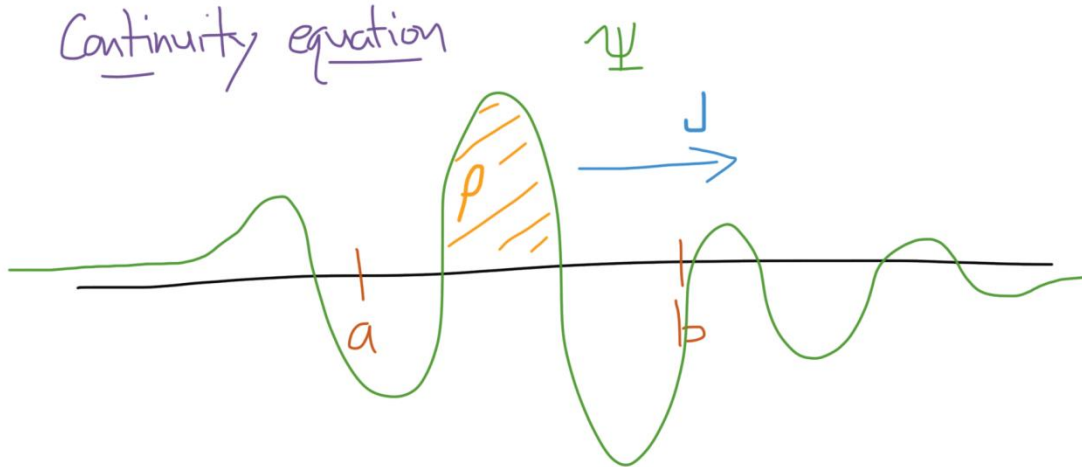$$\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0$$

Any charge that escapes a region of space must have passed through the boundary of that region. In a one-dimensional system

$$\frac{\partial J}{\partial x} + \frac{\partial \rho}{\partial t} = 0$$

Apply to the wavefunction. Consider an interval a to b on the real line. Any change in the density of the wavefunction in that interval must result from a density current.

$$d\rho = J(a) - J(b)$$

given the sign convention that J goes to the right.

What is it that's being conserved here? Conservation of probability. If amplitude translates in space – if the wavefunction is wiggling or a wave packet traveling – then the probability to find a particle must flow from one region to another. Probability of finding it somewhere remains 1, so the probability density has to increase in the neighboring interval by the amount it decreases in this interval right here.

Next up, the operator $\hat{H}$. In general, operators change the wavefunction. $\hat{H}$ is the time step operator. If the wavefunction is time dependent, e.g. if the wave is oscillating, then $\hat{H}$ updates the wavefunction to the next time step.

Here's the eye opener I mentioned at the beginning. Take the general Schrodinger equation.

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \left(\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x,t) \right) \Psi(x,t)$$

Remove the time dependence, i.e. consider a wavefunction in a fixed potential and psi itself a function (remember we're talking mathematical functions) depending only on position as the independent variable. Rewrite.

$$i\hbar \frac{\partial}{\partial t} \psi(x) = \left(\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right) \psi(x)$$

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Parentheses on the right represent total energy. Relabel the differential operator on the left and, voila

$$\hat{H}\psi(x) = E\psi(x)$$

Note that $\hat{H}$ is an operator and E is real. So this is an eigenvector / eigenvalue expression. We can solve the equation to find energy eigenvalues of the wavefunction, i.e. we can find the energy spectrum of a quantum system. Solve the differential equation. It is first order in time, so solution is pretty straightforward.

$$i\hbar\frac{\partial}{\partial t}\psi(x) = E\psi(x)$$

$$\psi(x,t) = e^{-iEt/\hbar}\,\psi(x)$$

Neat! We can calculate how the wavefunction evolves over time. Draw the wavefunction. Cartoon animate it to watch it change over time. Update the cartoon frames by that exponential in energy.

On to those energy eigenstates. The wavefunction will have particular energy solutions, $b_n\psi_n$ which are basis states in a vector space. So a general wavefunction can be expressed as

$$\psi = \sum_{i=1}^{n} b_i\psi_i$$

where n is the number of eigenstates, i.e. energy solutions, for that particular system. The summation above is the spectrum of the wavefunction, e.g. the energy states of a hydrogen atom. Interesting physics occurs in degenerate states, when more than two eigenstates have the same energy. More on that later.

The eigenstates are orthonormal, as expected in linear algebra. In mathspeak

$$\int \psi_m\psi_n\,dx = \delta_{mn}$$

All this is practically useful for calculating the general wavefunction

$$\Psi(x,t) = \sum_{j=1}^{n} b_j e^{-iE_j t/\hbar} \psi_j$$
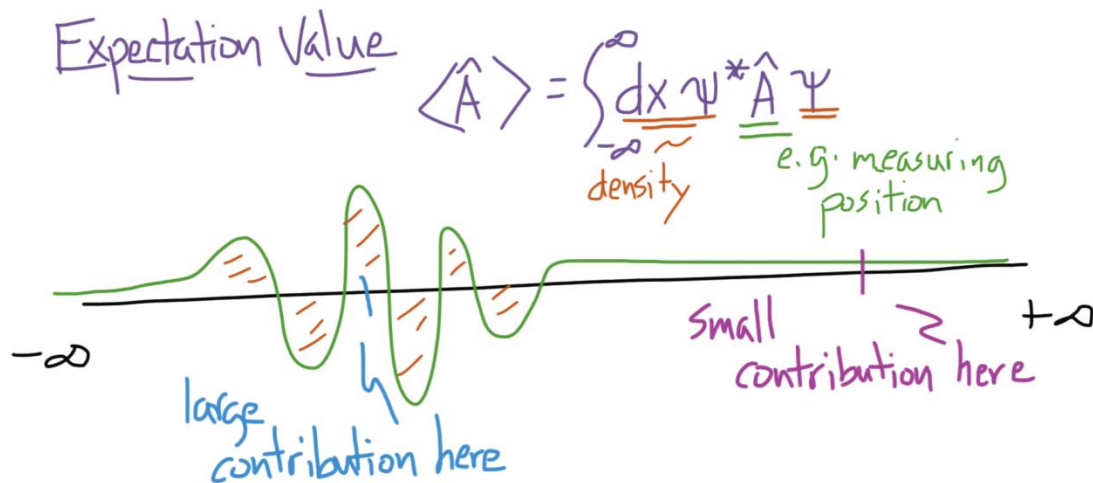
and for calculating the coefficients of the eigenstates

$$b_m = \int dx\, \psi_m{}^* \psi$$

Pay attention what's going on here. The summation above shows how to find the general wavefunction from the stationary function. All the time dependence is in the energy exponential. All the evolution, all the dynamics is in that energy function. The integral, finding coefficients, is all about orthogonality. Dot product (this really is just a dot product, in integral form) picks out the term in question. By orthogonality, all other products go to zero.

Finally, expectation value. Given a general time-independent operator, $\hat{A}$, what value can we expect on repeated / averaged measurements? Here 'tis.

$$\langle \hat{A} \rangle_\Psi (t) = \int_{-\infty}^{+\infty} \Psi^* \hat{A} \Psi \, dx$$

Real value on the left. Functions on the right. So the integral is a functional, converting a function to a number.

Expectation Value $\langle \hat{A} \rangle = \int_{-\infty}^{\infty} dx\, \psi^* \hat{A}\, \psi$

density

e.g. measuring position

large contribution here

Small contribution here

$-\infty$

$+\infty$

Think about that a minute. Suppose you're trying to find a particle's position, looking for the expectation value of the position operator. Well, that argument in the integral is teasing out the likeliest position from the probability density, the product of those Psi's. The integral is finding average value over an infinite range where total probability is one, so there's no need for the usual $1/(b-a)$ coefficient out front. Particle is most likely to be found where the probability density is greatest.

We'll see later that we can interpret the integral as sum of the projections of the rotated state vector. The operator transforms the wavefunction. That's what matrices = operators do. Rotate or stretch vectors. Assuming the wavefunction is normalized, then the dot product of the vector with its transformed self gives you the projection, how much of that wavefunction you can expect to find with that observation. Projections on vectors. Observables and how much you can expect to observe.

Lecture 2: bound states

Prof. Zwiebach starts out with theorems about bound states, i.e. states that go to zero at $\pm\infty$.

- They're non-degenerate. No duplications of states at the same energy.
- They're real.
- And they're either even or odd functions.
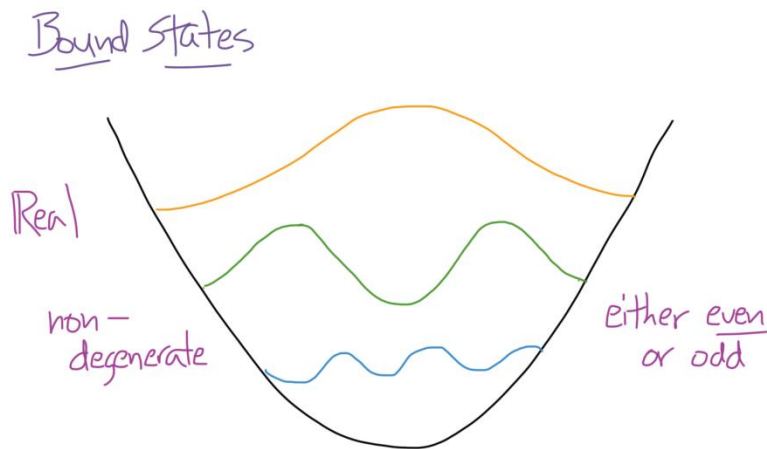
The lecture includes corollaries and strategies for the proofs. See the course notes for details.

Lecture 3: position and momentum

Position and momentum are observables, i.e. they have a physical instantiation that we can measure. We can think of them also as different bases, different vector spaces describing a physical system. Prof. Zwiebach introduces the essential linear algebra.

You can switch from one basis to the other using Fourier transforms.

$$\psi(x) = \int_{-\infty}^{\infty} dp \, e^{ipx/\hbar} \, \tilde{\psi}(p)$$

and

$$\tilde{\psi}(p) = \int_{-\infty}^{\infty} dx \, e^{-ipx/\hbar} \, \psi(x)$$

Note that the momentum operator acting on the x-basis wavefunction gives the associated eigenvalue relations, and vice versa for the p-basis wavefunction:

$$\int_{-\infty}^{\infty} dp \, e^{ipx/\hbar} \, \tilde{\psi}(p) \cong \sum_{j=1}^{N} e^{ip_j x/\hbar} \, \tilde{\psi}(p_j)$$
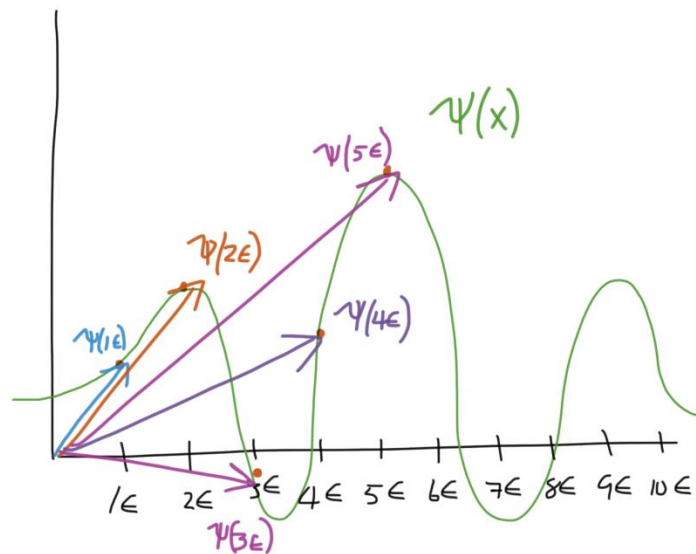
so

$$\hat{p}\psi(x) = -i\hbar\frac{\partial}{\partial x}\psi(x) = -i\hbar\frac{\partial}{\partial x}\sum_{j=1}^{N} e^{ip_j x/\hbar}\,\tilde{\psi}(p_j) = p\psi(x)$$

as expected. It's all in that exponential.

One of the neat things I learned in this lecture is how to think of the wavefunction as a vector. Draw a one-dimensional $\psi(x)$. Parse out the function over intervals ϵ. The wavefunction has a value at each interval. Voila! A vector!

$$\psi(x) = \begin{bmatrix} \psi(0) \\ \psi(\epsilon) \\ \psi(2\epsilon) \\ \psi(3\epsilon) \\ \vdots \end{bmatrix}$$

And the position operator is a matrix. Given

$$\hat{x}\psi(x) = x\psi(x)$$

Translate to linear algebra

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$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & 2\epsilon & \\ \vdots & 0 & 0 & 3\epsilon \end{bmatrix} \begin{bmatrix} \psi(0) \\ \psi(\epsilon) \\ \psi(2\epsilon) \\ \psi(3\epsilon) \\ \vdots \end{bmatrix} = x\psi(x)$$

Makes sense! And now I appreciate why the wavefunctions sit in such a huge (Hilbert) vector space!

The rest of the lecture introduces the Stern-Gerlach experiment. Key is understanding magnetic moment and how a divergent external B field can separate spin-up from spin-down.
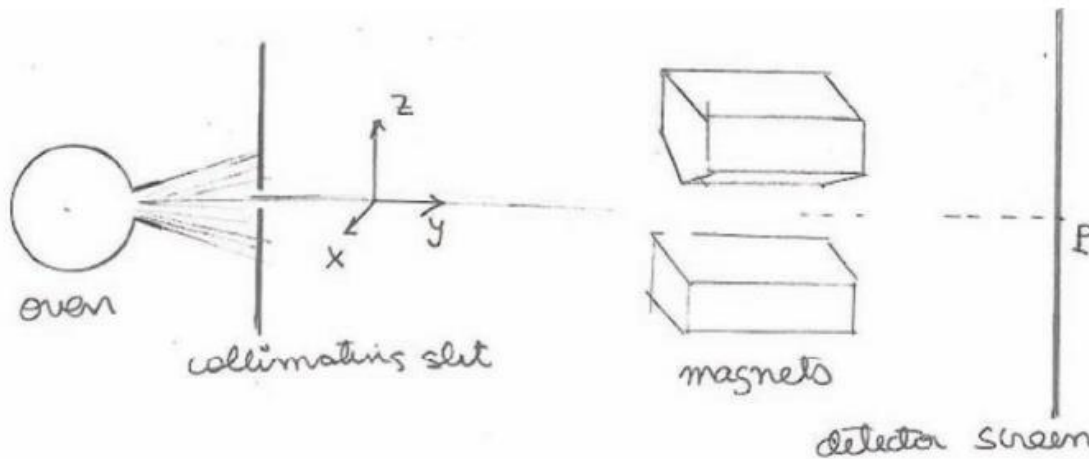

Figure. Stern-Gerlach apparatus. Collimated beam of ionized silver atoms traverses gradient of magnetic field, which separates spin up from spin down. Credit Prof. Barton Zwiebach, MIT OCW Physics 8.05.
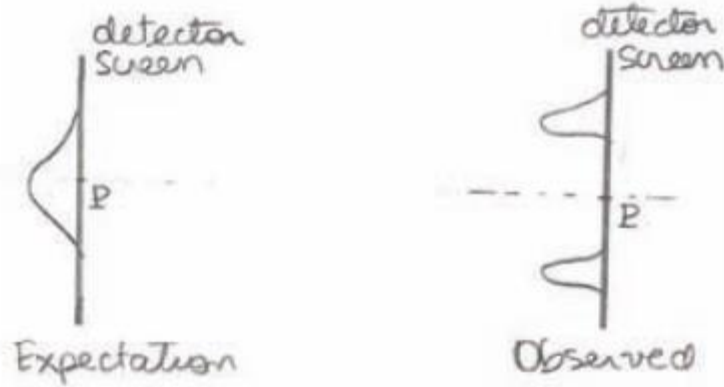
Figure. Contrary to prediction of classical electromagnetism, electrons are only detected in one or the other of two states. Credit Prof. Barton Zwiebach, MIT OCW Physics 8.05.
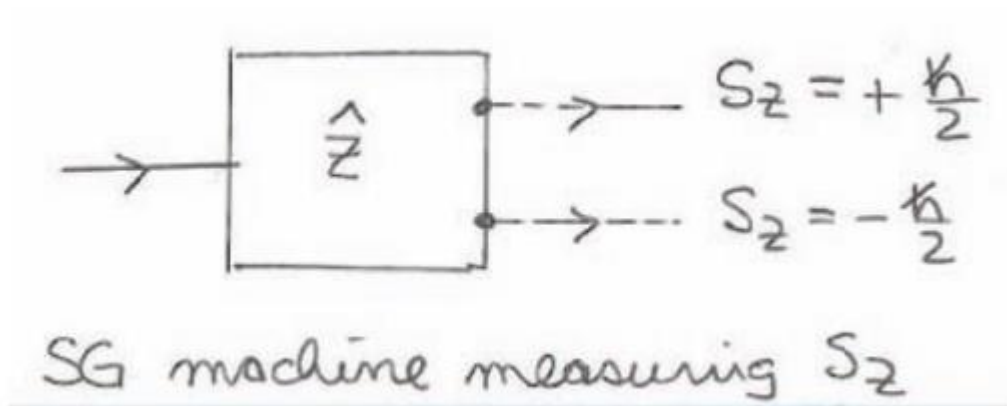

Figure: Diagram of Stern-Gerlach results. Credit Prof. Barton Zwiebach, MIT OCW Physics 8.05.
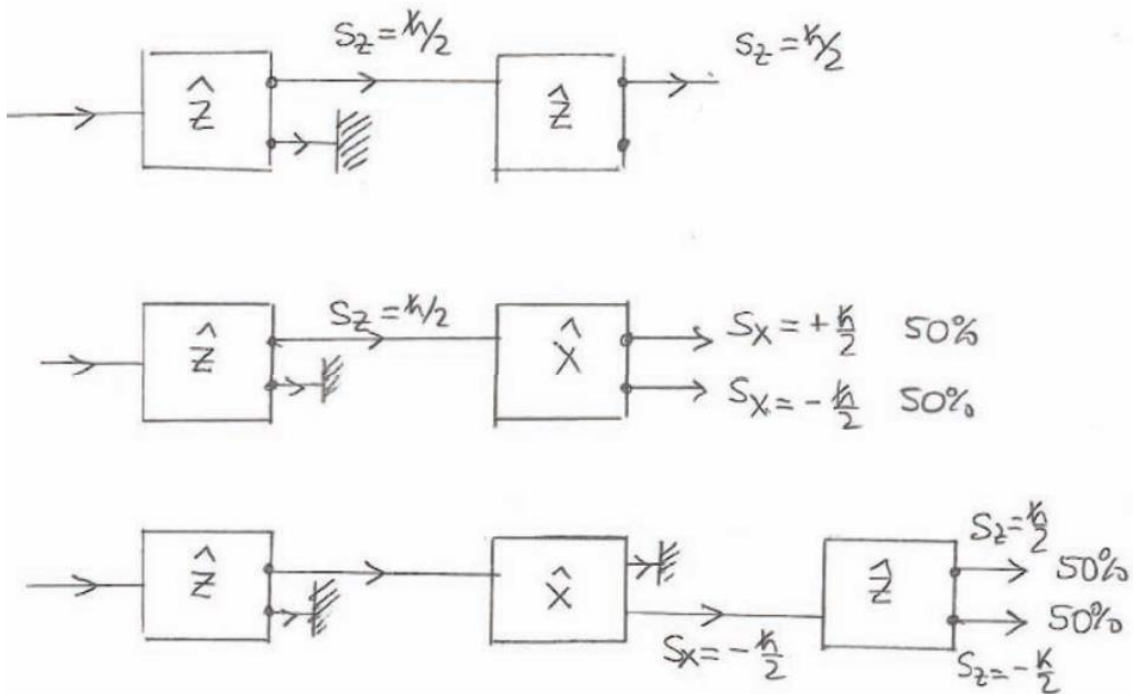
Figure: Series of Stern-Gerlach apparatus at different orientations. This is the heart of quantum mechanics! Credit Prof. Barton Zwiebach, MIT OCW Physics 8.05.

See Zwiebach's Lecture 3 for thorough discussion of spin one-half and the spin operators. I thought I had recorded those notes, but they've disappeared somewhere.

Lectures 4–7: Linear algebra

I learned a whole bunch here. Zwiebach was away for a couple lectures, so Aram Harrow and William Detbold filled in. They zoomed through all the essentials. What a great review!

They based their presentation on Axler's text, *Linear Algebra Done Right*. I believe that title. Here are the main take-aways and what now makes sense that didn't before (or that I just assumed I understood but really didn't).

It's all about vector spaces and their properties. Components of vector spaces are fields and vectors, properties collected in CANNDII. Fields for our practical purposes are the reals and complex numbers. Vectors are of many sorts: polynomials, lists, etc. Addition of vectors commutes. Vector addition is associative. There is a null (zero) vector for multiplication. There is a negative (inverse) for addition. Multiplication is distributive. And identities exist for both addition and multiplication.

Hilbert space is a complex vector space that includes an inner product. Inner product is an operation that gives a field value when two vectors are multiplied; best example is the dot product. All physics occurs in Hilbert space. I think the physical implication here is that physical space requires a measure of distance, and that's given by the inner product.

A subspace U of vector space V contains the field and vectors $\vec{u}$ such that $U \subset V$ and U itself is a vector space by all the definitions. Now it must be the case for some combination of $U's$ that

$$U_1 \oplus U_2 \oplus \ldots U_n = V$$

where the $\oplus$ mean the 'direct sum.' We're adding subspaces to build the larger vector space.

Now consider. Suppose each U comprises linearly independent basis vectors that span U. Then the direct sum spans V. The direct sum forms a basis for V. That's the definition of a basis: a set of linearly independent vectors that spans the vector space.

Operators are maps that transform vectors in a vector space V to other vectors in that same vector space. Operators can be represented by matrices, but matrices are more narrowly defined as operators in a given basis. The same matrix produces different results in different bases. Operators are functions and obey the distributive laws and linearity, but they do not necessarily commute. The commutation part is interesting; the commutator is a kind of eigenvalue relation. Suppose operator $R = x$ and operator $S = \partial/\partial x$. Then, acting on the polynomial $p = x^n$

$$[S, R]p = Ip$$

I is like an eigenvalue here. Just like $i\hbar$ acts like an eigval in $[x, p] = i\hbar$

The dimension of a vector space

$$\dim V = \dim null(T) + \dim range(T)$$

This is the Fundamental Theorem of vector spaces. $null(T)$ are all those vectors that the operator T takes to zero (zero vector). $range(T)$ are all the (surjective) transformations Tv otherwise filling the vector space V. Injective means an operator maps one-to-one in the vector space. Surjective means the map fills the whole vector space.

Eigenvectors and eigenvalues of an operator are in the subspace of a vector space such that the operator acting on an eigenvector returns another vector in that subspace.

$$T\vec{u} = \lambda\vec{u}$$

I've sure seen that before, but it makes more sense in the context of the spaces. Axler rules!

Which leads to another useful theorem and perspective on the spaces. The eigen-subspace plus the subspace orthogonal to the eigens fills the next higher dimension.

$$V = U + U^{\perp}$$

You can see that with the 2d $x - y$ plane, spanned by e_x and e_y. Add the perpendicular e_z and you've got $\mathbb{R}^3$.

Maybe the biggest 'aha' of these lectures was Prof. Zwiebach's explanation of Dirac's notation. It derives from the inner product in complex space. The ket is a good ol' regular vector. (Remember, whether a vector space is real or complex depends on the field, not the vectors.) The bra on the other hand (and this was the 'aha') is a map. It maps the ket to a complex value in the field. (Which is what happens with an inner product.)

The bra's compose an injective dual space to the kets. i.e. the bra is unique to the dual ket. Operations apply the rules for complex numbers.

$$\langle a|b \rangle = \langle b|a \rangle^*$$

$$\langle a|\beta_1 b_1 + \beta_2 b_2 \rangle = \langle \beta_1 b_1 + \beta_2 b_2|a \rangle^* = \langle \beta_1 b_1|a \rangle^* + \langle \beta_2 b_2|a \rangle^* = \beta_1{}^* \langle a|b_1 \rangle + \beta_2{}^* \langle a|b_2 \rangle$$

I skipped a step at the last, swapping bra and ket again. That's why the complex star disappears on the last brackets. But you get the idea.

Of course, you can represent the bra's as row vectors; inner product then just becomes row times column vector multiplication. Handy!

References are in the 8.05 notes, all you need to know, and of course in Axler. What great resources those are.

Detbold and Harrow look like kids, but they are on the frontiers themselves. Harrow works in quantum computers and information theory. He's a student of Chuang's. Detbold runs numerical simulations of strong interactions, including conditions at the cores of neutron stars. Neat stuff!

Lecture 8: bra's and ket's and operators

Now we get to the really powerful payoffs. Look at what the bra and ket logic can do.

Take for example the definition of the dagger operator and its operations. A couple useful theorems for operators pop out.

$$\langle T^\dagger u | v \rangle = \langle u | Tv \rangle$$

If we're in a real vector space, then $T^\dagger = (T^*)'$. That's one of the theorems. And you can use it both ways – find the T's if you're in a real vector space or prove you're working with a real vector space if you know the T's. Also note – just formalism – that $\langle u | Tv \rangle = \langle u | T | v \rangle$ and, most important for practical purposes, we can treat the T's as matrices.

The other handy theorem:

$$\langle T^\dagger u | v \rangle = \langle v | Tu \rangle^*$$

Proofs for all these rest on index manipulation. I had it all clear in my head, but then it went fuzzy.

Anyway, them's important operators, the T's. I think they'll have a lot to do with projections and measurement, expectation values and such truck.

Lecture 9: bra's and kets (cont'd)

Ha! Told you so. That there expression $\langle u | Tv \rangle = \langle u | T | v \rangle$ is an expectation value! Zwiebach let it slip in passing.

Main arguments in this lecture are extensions of the definition of a Hermitian operator. $T^\dagger$ is the adjoint, as defined by the bracket relations above, while $T^\dagger = T$ defines Hermitian adjoint. And for Hermitian operators all kinds of wonderful properties follow. For example, the eigenvalues of Hermitian operators must be real. So they can describe measurements. And the eigenvectors of different eigenvalues must be orthogonal. Proofs are pretty slick.

Given T Hermitian and $Tv = \lambda v$ eigens. Start with bracket $\langle v | Tv \rangle$ and work both directions.

$$\lambda \langle v | v \rangle = \langle v | \lambda v \rangle = \langle v | Tv \rangle = \langle T^\dagger v | v \rangle = \langle Tv | v \rangle = \langle \lambda v | v \rangle = \lambda^* \langle v | v \rangle$$

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Expressions on the two ends are equal, so the lambdas must be real for $\lambda = \lambda^*$. Done! All that work finding $T^\dagger$ pays off!

What about orthogonality of eigenvectors? Here 'tis. Same kind of math manipulation.

Given two different eigenvalues on the same Hermitian operator:

$$\lambda_1\langle v_1|v_2\rangle = \langle \lambda_1 v_1|v_2\rangle = \langle Tv_1|v_2\rangle = \langle v_1|Tv_2\rangle = \langle v_1|\lambda_2 v_2\rangle = \lambda_2\langle v_1|v_2\rangle$$

So $(\lambda_1 - \lambda_2)\langle v_1|v_2\rangle = 0$

But we've stipulated the eigenvalues are different, so $\langle v_1|v_2\rangle = 0$. The eigenvectors must be orthogonal!

Onward to unitary operators. By definition a unitary operator is Hermitian and preserves the norm of any vector in the vector space.

$$|Uv| = |v|$$

From this, since U is Hermitian,

$$\left|U^\dagger Uv\right| = |Uv| = |v|$$

So $U^\dagger U = I$. The identity. We'll be using that.

On to bra's. Now here's new. We've been used to linear algebra

$$|av\rangle = a|v\rangle$$

Fine and dandy. Linearity on a vector space. But look what happens when we start talking about position and momentum, non-determinate variables. That is, position, a particle state, could be anywhere in an infinite dimensional Hilbert space. Proper interpretation of the bra, then, is that it represents the state of a particle, not the vector position of the particle. $|x\rangle$ represents a particle at position x. It is the state of the particle being at x. So

$$|ax\rangle \neq a|x\rangle$$

Left side is a particle in the state of being at position ax. Right side a is the amplitude of a particle in the state of being at position x. There's a difference! Similarly $|\vec{x}\rangle$ is the state in

which a particle is sitting at 3-d vector position $\vec{x}$. The bra itself is not the vector coordinate position. Subtle, eh? But make sense when you think about it.

Lecture 10: Uncertainty

Here's the absolute coolest thing ever. Pythagoras rules!

Question is: how do you figure the uncertainty in an operator? Suppose you're trying to find the value of an observable. Operate on the wavefunction. How certain are you of the result?

Here's how you find the uncertainty of a unitary operator $\hat{A}$ acting on a normalized wavefunction ψ .

$$\Delta A|\Psi\rangle = \left(A - \langle A\rangle I\right)|\Psi\rangle$$
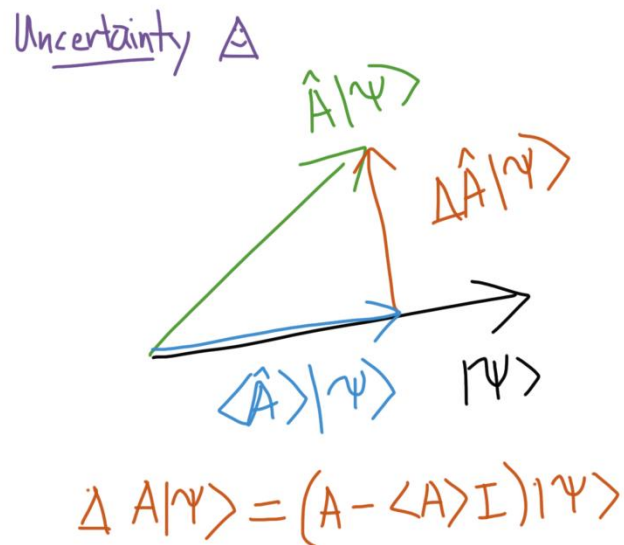
A rotates $|\psi\rangle$. First calculate the projection of $A|\psi\rangle$ onto the eigenvector $|\psi\rangle$. That will give the amplitude to find the transformed vector in that eigenstate.

Projection. It's an operator. It casts the shadow of a state vector onto a basis vector and thereby represents the component of that basis in that particular state. It's a simple calculation. Using the $|\psi\rangle$ basis as an example,

$$P_{|\psi\rangle} = |\psi\rangle\langle\psi|$$

That's it! Note that that the projection is a matrix, so a likely operator sure enough. And look here. Projection of $A|\psi\rangle$ onto the basis gives the expectation value in that basis.

$$|\psi\rangle\langle\psi|A|\psi\rangle = |\psi\rangle\langle A\rangle = \langle A\rangle|\psi\rangle$$

All that's legal because the expectation value $\langle A\rangle$ is a real number (because the operator is Hermitian).

Good enough. Expectation value gives the amplitude of the transformed vector on that basis.

And, methinks, that there explains a lot! Look back at the Dirac definition of the wavefunction.

$$\psi = \int dx \; |x\rangle\langle x|\psi\rangle$$

That's the projection of psi onto the x-basis. Sum over all x and you've got the state vector = the wavefunction! Remember, x is a non-determinable basis. It's the Hilbert space of all x. Each position itself serves as a basis. So the definition says the wavefunction is the sum of all the projections of the state vector onto all the bases. Same idea as a vector in three-space is the sum of its $x, y,$ and z components. Makes sense!

Projections onto basis states are components. Keep that in mind.

But back to the uncertainty. Almost done. Look at the triangle. Because all the vectors are unit vectors, projection is cosine of the angle between $A|\psi\rangle$ and $|\psi\rangle$. Previous of Zwiebach's calculations found that the uncertainty

$$\Delta A|\psi\rangle = A|\psi\rangle - \langle A\rangle I|\psi\rangle$$

You have to have the I in there because we're dealing with matrix operators. But take a look at the figure. $\Delta A|\psi\rangle$ is the sine leg of that Pythagorean triangle. That's the uncertainty, expressed as a vector. And Pythagoras immediately gives us the most useful relation for calculating uncertainties.

$$(\Delta A)^2 \geq \langle A^2\rangle - \langle A\rangle^2$$

It's all from the triangle. But the $\langle A^2\rangle$ isn't obvious. Here's the derivation. Remember the state vector is imaginary. So

$$(A|\psi\rangle)^2 = (\langle\psi|A)(A|\psi\rangle) = \langle\psi|A^2|\psi\rangle = \langle A^2\rangle$$

Pretty neat! There's a whole lot more algebra in the formal proof, but it's all right there in that vector diagram. And a whole lot easier to remember.

Note also the relation to statistics. That 'mean of the square minus the square of the mean' is the statistical variance. Variance. How widely dispersed are the elements in the sample. Applied to operators and states, I suppose it could represent the variation you'd expect if you performed a whole series of experiments on particles all prepared in the same initial state.

Lecture 10: the uncertainty principle

Variance and uncertainty, related by experiment. The uncertainty in repeatable experiments is the variance around the mean, the expectation value. Measure spin along the x-axis of zillions of electrons prepared with spin $|z+\rangle \equiv |+\rangle = |\uparrow\rangle$. We want to know the variance of our experimental results, our observations, our observable. What is the uncertainty in s_x ?

Think about that again. Prepare spin up along z . Measure along x with good ol' Stern-Gerlach. Electron spin angular momentum is $\hbar/2$. Expectation value is zero (because it's equally likely we'll measure $|x+\rangle$ or $|x-\rangle$. We want to know how much variance there will be around the expectation value. Well, duh. Spin is quantized. There are only two possible outcomes to our measurements, $+\hbar/2$ or $-\hbar/2$. Variance is $\hbar/2$. Uncertainty of the outcome is $\hbar/2$. Uncertainty in the observable s_x is $\hbar/2$. Done.

Let's see if the math works. We want to find the variance in s_x given state $|z+\rangle$. Use the formula.

$$(\Delta s_x)^2 \geq \langle s_x{}^2 \rangle - \langle s_x \rangle^2$$

Convert to matrix representation to calculate those expectation values.

$$\langle s_x \rangle = \langle +|s_x|+\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

$$\langle s_x{}^2 \rangle = \langle +|s_x{}^2|+\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\frac{\hbar}{2}\right)^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left(\frac{\hbar}{2}\right)^2$$

Put it all together,

$$(\Delta s_x)^2 \geq \left(\frac{\hbar}{2}\right)^2$$

$$\Delta s_x \geq \frac{\hbar}{2}$$

As expected! The logic works. Variance = uncertainty lower bound as expected from the quantization of spin.

Okay. How about those famous uncertainty principles? $\Delta x \Delta p \geq \frac{\hbar}{2}$ and $\Delta E \Delta t \geq \frac{\hbar}{2}$? Where did they come from? How can we understand them? And how can we figure out the minimum of uncertainties, the minimum variance in our experimental results?

Jumping right to the general formula,

$$\Delta A \Delta B \geq \left(\langle \psi | \frac{1}{2i} [A, B] | \psi \rangle\right)$$

where A and B are Hermitian operators. Check that out with the known

$$\Delta x \Delta p \geq \left(\langle \psi | \frac{1}{2i} [x, p] | \psi \rangle\right) = \left(\langle \psi | \frac{1}{2i} i\hbar | \psi \rangle\right) = \frac{\hbar}{2}$$

As expected. Note that the uncertainty is real. Now where did that general formula come from? Back to basics. We need to figure out that commutator in the expectation value. Begin with the Schwarz inequality. Product of norms of the two sides of a triangle is always greater than or equal to the product of the projection of a vector and its projector.

Schwarz Inequality

$$\vec{u}$$

$$P_v \vec{u} = |v\rangle\langle v|u\rangle$$

$$\vec{v}$$
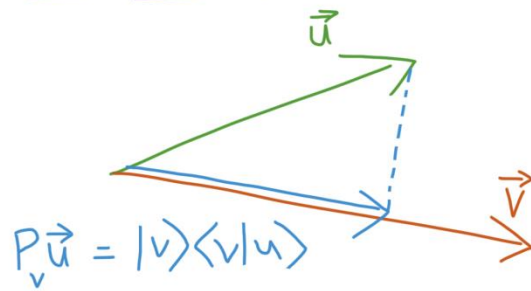
$$(u,u)(v,v) \geq (u,v)^2$$
$$(u,v) = \vec{v} P_v \vec{u}$$

Apply that to our geometric interpretation of the uncertainty.

Uncertainty △

$$\hat{A}|\Psi\rangle$$

$$\Delta\hat{A}|\Psi\rangle$$

$$\langle\hat{A}\rangle|\Psi\rangle \qquad |\Psi\rangle$$
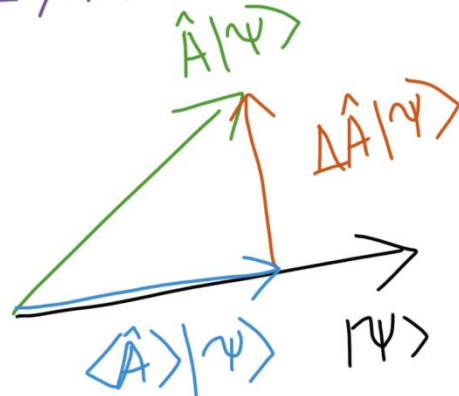
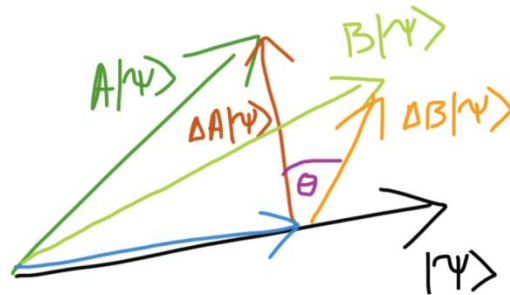$$\Delta A|\Psi\rangle = \left(A - \langle A\rangle I\right)|\Psi\rangle$$

$$\Delta A|\psi\rangle = A|\psi\rangle - \langle A\rangle I|\psi\rangle$$

Add a second uncertainty to the mix.

$$\Delta B|\psi\rangle = B|\psi\rangle - \langle B\rangle I|\psi\rangle$$

Look at the geometry.

Uncertainty Principle

$$(\Delta A)^2(\Delta B)^2 \geq \left(\langle \Delta A|\Delta B\rangle\right)^2$$

The operators A and B are working in two different bases. They operate on the same state, $|\psi\rangle$, but in different basis states. Position and momentum, say. Imagine the state $B|\psi\rangle$ is projecting out of the plane of the paper in that different basis, rotating its uncertainty $\Delta B|\psi\rangle$ along with it. We're interested in calculating $(\Delta A)(\Delta B)$. We'll use Schwarz. In preview, though, we'll be interested the projection of ΔA onto ΔB. That tells all.

We're out to prove

$$\Delta A \Delta B \geq \left|\langle\psi|\frac{1}{2i}[A,B]|\psi\rangle\right|$$

Convert to the Schwarz form

$$\Delta A^2 \Delta B^2 \geq \left(\langle\psi|\frac{1}{2i}[A,B]|\psi\rangle\right)^2$$

Let

$$A|\psi\rangle - \langle A\rangle I|\psi\rangle \equiv f|\psi\rangle$$

and

$$B|\psi\rangle - \langle B\rangle I|\psi\rangle \equiv g|\psi\rangle$$

Schwarz gives

$$\langle f|f\rangle\langle g|g\rangle \geq (\langle f|g\rangle)^2$$

We focus on that rhs of the uncertainty inequality. First some basic, straight off the Argand plane complex geometry.

$$(\langle f|g\rangle)^2 = Im(\langle f|g\rangle)^2 + Re(\langle f|g\rangle)^2$$

Left side is complex, imaginary and real parts. Pythagoras separates them. Let's see where that leads.

$$Im(\langle f|g\rangle)^2 = \frac{1}{2i}(\langle f|g\rangle - \langle g|f\rangle)$$

That's just another manipulation of the complex. $\frac{1}{2i}(a+bi) - (a-bi) = b$, the imaginary component of the complex z .

Similarly,

$$Re(\langle f|g\rangle)^2 = \frac{1}{2}(\langle f|g\rangle + \langle g|f\rangle)$$

Now it turns out the real component usually doesn't affect the Schwarz inequality. What matters is the imaginary component. That's because the real term is always positive and always, well, real. It's that differential in the imaginary component that's tracking uncertainty. It's the imaginary component that takes us out of the real plane and into the variance due to measurement in different bases. A measurement with meter sticks vs. a measurement with stopwatches, say. What's the variance in outcome when you have to consider both the error in your meter stick and the error in the stopwatch. Measure a meter a bit too long and a time too short, then measure a meter a bit too short and a time too long and you've multiplied your variance. That's what's included out there in the angle theta between the uncertainty vectors. At

least I think so. Anyway, we'll take Zwiebach at his word and focus on that imaginary component. Time to plug the operators back in.

$$Im(\langle f|g\rangle)^2 = \frac{1}{2i}(\langle\psi|\,(A - \langle A\rangle)(B - \langle B\rangle)|\psi\rangle) - (\langle\psi|\,(B - \langle B\rangle)(A - \langle A\rangle)|\psi\rangle)$$

Do the algebra.

$$\frac{1}{2i}(\langle\psi|\,(A - \langle A\rangle)(B - \langle B\rangle)|\psi\rangle) - (\langle\psi|\,(B - \langle B\rangle)(A - \langle A\rangle)|\psi\rangle)$$

$$= \frac{1}{2i}(\langle\psi|\,(AB - A\langle B\rangle - \langle A\rangle B + \langle A\rangle\langle B\rangle)|\psi\rangle) - (\langle\psi|\,(BA - B\langle A\rangle - \langle B\rangle A + \langle B\rangle\langle A\rangle)|\psi\rangle)$$

Good. Now take a look. Those are expectation values with mixed terms, scalars (other expectation values), and operators for which the bracketing psi's will calculate expectation values. For example pull out that first $A\langle B\rangle$ term. We're working in linear vector spaces after all. We can separate terms.

$$\langle\psi|\,(-A\langle B\rangle)|\psi\rangle = -\langle B\rangle^*\langle\psi|\,(A)|\psi\rangle = -\langle B\rangle^*\langle A\rangle$$

It's the product of two expectation values, itself a number.

If you simplify all those manipulations inside the original brackets,

$$Im(\langle f|g\rangle)^2 = \frac{1}{2i}(\langle\psi|AB - \langle A\rangle\langle B\rangle\,|\psi\rangle) - (\langle\psi|BA - \langle B\rangle\langle A\rangle\,|\psi\rangle)$$

By the rules, collect the terms in the brackets.

$$Im(\langle f|g\rangle)^2 = \frac{1}{2i}(\langle\psi|AB - \langle A\rangle\langle B\rangle - BA + \langle B\rangle\langle A\rangle\,|\psi\rangle) = \frac{1}{2i}(\langle\psi|\,[A,B]\,|\psi\rangle)$$

As promised! Pretty slick! Look how that commutator is telling us the uncertainty. It's the variance between measurement in different bases.

Next up: derivation of the Energy-time uncertainty relation. Now we've got the general formula; all we have to do is plug in likely operators.

Start with the time-dependent SE and a dummy operator, Q, that is time independent.

$$ i\hbar \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle $$

Bracket the commutator of H with Q.

$$ \langle\psi| [H, Q] |\psi\rangle = \langle\psi| HQ - QH |\psi\rangle = \langle\psi| HQ |\psi\rangle - \langle\psi| QH |\psi\rangle $$

$$ = \langle H\psi| Q |\psi\rangle - \langle\psi| Q |H\psi\rangle = -i\hbar \frac{\partial}{\partial t} \langle\psi| Q |\psi\rangle - i\hbar \frac{\partial}{\partial t} \langle\psi| Q |\psi\rangle = -2i\hbar \frac{\partial}{\partial t} \langle Q \rangle $$

Let's check that. Work backward.

$$ \frac{\partial}{\partial t} \langle Q \rangle = \frac{\partial}{\partial t} \langle\psi| Q |\psi\rangle = \left(\frac{\partial}{\partial t}\psi \right| Q\psi\rangle + \langle\psi| Q \frac{\partial}{\partial t}\psi \right) = \frac{1}{i\hbar} (\langle H\psi| Q |\psi\rangle - \langle\psi| Q |H\psi\rangle) $$

$$ = \frac{1}{i\hbar} (\langle\psi| HQ |\psi\rangle - \langle\psi| QH |\psi\rangle) = \frac{1}{i\hbar} (\langle\psi| HQ - QH |\psi\rangle) $$

$$ = \frac{1}{i\hbar} \langle\psi| [H, Q] |\psi\rangle $$

Looks promising, but I've lost some factors in there somewhere.

Note that $\langle Q \rangle$ is time dependent. Plug back in to Schwarz.

$$ (\Delta H)^2 (\Delta Q)^2 \geq \left(\langle\psi| \frac{1}{2i} [H, Q] |\psi\rangle \right)^2 = \left(-\hbar \frac{\partial}{\partial t} \langle Q \rangle \right)^2 $$

$$ \rightarrow \Delta H \Delta Q = \left| -\hbar \frac{\partial}{\partial t} \langle Q \rangle \right| $$

Now identify a time interval

$$ \Delta t = \frac{\Delta Q}{\frac{\partial \langle Q \rangle}{\partial t}} $$

That's the time interval required for the operator Q to change by amount ΔQ.

Done! Well, factor of two or so. $\Delta H \Delta t = \hbar$. Identify ΔH with its eigenvalue and we've got it.

$$\Delta E \Delta t = \hbar$$

Good enough.

<u>Lecture 11</u>: Uncertainty and the spectral theorem

Professor Z. checks out the position-momentum uncertainty with an example,

$$H = \frac{p^2}{2m} + x^4$$

First convert to operators, then carry on. Gives pretty good prediction for the experimental values. (Not sure where he gets those experimental data.) See his lecture notes.

On to diagonalization. If an operator has a complete set of eigenvectors on a vector space, then you can diagonalize the operator, eigenvalues along the diagonal with corresponding eigenvectors. Better yet, you can find an orthonormal diagonalization: eigenvalues along the diagonal and orthonormal eigenvectors.

Here are the necessary transformations. Given

$$Tv = \lambda v$$

with operator $T \in \mathcal{L}(V)$ and $v = \{v_i\}$ a set of basis vectors on V. Then there exists an operator A such that $Av_i = u_i$, $u = \{v_i\}$ another set of basis vectors on V and

$$Tu = \lambda u$$

with T diagonal. That's possible because you can choose v orthonormal so that

$$u_i = A_{ik} v_k = \begin{bmatrix} A_{1k} \\ A_{2k} \\ \vdots \\ A_{nk} \end{bmatrix}$$

because

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \qquad etc.$$

Better yet, you start with an orthonormal v and choose a unitary operator A such that u also is an orthonormal basis. That makes T as simple as possible: a diagonal matrix, all the eigenvalues along the diagonal, with orthonormal eigenvectors.

How find that T? Assume v orthonormal and A unitary as per above.

$$Tv = \lambda v \ \rightarrow \ TAv = \lambda Av \ \rightarrow \ A'TAv = A'\lambda Av = \lambda v$$

$$A'Tu = \lambda v$$

There's your diagonalized matrix, $A'T$.

I've skipped over some subscripts. Those lambda's, of course, are different in the different bases. But onward.

Main idea in this lecture, and one of the key ideas for all Heisenberg-Dirac QM is the spectral theorem. If an operator T in vector space V has eigenvectors that form an orthonormal basis in V, then T must be a normal operator. By definition a normal operator has the property that

$$[T^\dagger, T] = 0$$

Normal operators include Hermitian and unitary operators. They're the ones we need for quantum mechanics.

The rest of the lecture involves proofs for the spectral theorem and various properties of diagonalization. I guess the practical takeaway for all this is if you can diagonalize an operator, then you immediately see it's spectrum; all the eigenvalues sit there along the diagonal. And all the eigenvectors are just 1's in the proper slot on the unit vectors.

The other idea here is figuring out under what conditions you can simultaneously diagonalize two operators. Key requirement is that the two operators must have bases of the same dimension in the same vector space. Then you can relate one to the other by converting the bases. Diagonalize one and you know how to diagonalize the other by change of basis.

Complications enter when you have a degenerate operator. Same energy, say, for several different eigenstates. You want to be able to find out what makes those states different if it's not the energy. That's the whole point of the simultaneous diagonalization of different operators. Represent one operator in terms of the other and you figure out what distinguishes them.

The matrix math is interesting. Turns out that the diagonalization of a degenerate operator gives you a block diagonal matrix. Each block represents the subspace of eigen's with the same energy. That's the conceptually important point. Degenerate eigenstates mean you have several vectors (states) composing a vector subspace all with the same energy. Each energy is a subspace of the larger vector space. Add up all those subspaces, a direct sum, and you get the entire vector space. Anyway, once you've figured out the blocks you can diagonalize the individual blocks with a new set of matrix transformations inherent to that subspace. There's a whole lot of matrix calculations going on, but do-able.

Lecture 12: Harmonic oscillator and quantum dynamics

Harmonic oscillator. Again? For heaven's sake, why?

Well, we've got new tools and we can use them to pry further under the hood. Normal operators and the spectral theorem provide new understanding. Nuts and bolts. We get to see more of the blueprints, how quantum mechanics is put together.

Start with the usual

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

Factor. Purpose here is to produce a form $H \sim V^\dagger V$ so we can apply the spectral theorem.

$$H = \frac{1}{2}m\omega^2\left(\hat{x} - \frac{\hat{p}}{m\omega}\right)\left(\hat{x} + \frac{\hat{p}}{m\omega}\right) + \frac{\hbar}{2}\omega$$

Last term there, remember, results from the cross terms between factors and their commutation relations.

Okay. So what? Well, those parentheses are normal operators! Let

$$\left(\hat{x} - \frac{\hat{p}}{m\omega}\right) = V^\dagger \text{ and } \left(\hat{x} + \frac{\hat{p}}{m\omega}\right) = V$$

Normal operators. Their product gives a Hermitian operator, H. Their (orthonormal) eigenvectors provide a basis for their complex (Hilbert) vector space, so they are diagonalizable and their eigenvalues sit on the diagonals. We just have to read off the diagonals to find the spectrum of the harmonic oscillator. Can't get any spiffier than that!

On to the nit and grit and an interesting aha! Define

$$\hat{a} \equiv \sqrt{\frac{m\omega}{2\hbar}} V \text{ and } \hat{a}^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}} V^\dagger$$

Then

$$H = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$$

Tidy! Even tidier a number operator $N \equiv \hat{a}^\dagger \hat{a}$. It's going to tell us which energy state we're in. It's an integer. With that

$$H = \hbar\omega \left(N + \frac{1}{2} \right)$$

Some consequences of importance:

$$[\hat{a}^\dagger, \hat{a}] = -1 \text{ } and \text{ } [\hat{a}, \hat{a}^\dagger] = 1$$

Check it out. With that, now here's cool and illustrative. Remember working on the Energy-time uncertainty we found that

$$\frac{\partial}{\partial t} \langle Q \rangle = \frac{i}{\hbar} [H, Q]$$

Where Q is any general operator. Flip the other way, the commutator of the Hamiltonian on an operator is the time derivative of that operator with that quantum factor coefficient $\frac{i}{\hbar}$. Well, looky here.

$$\frac{i}{\hbar} [H, \hat{a}] = \frac{\partial}{\partial t} \hat{a} = \frac{i}{\hbar} (-\hbar\omega\hat{a}) = -i\omega\hat{a}$$

The sequence of equations comes from

$$[H, \hat{a}] = -\hbar\omega\hat{a}$$

You can derive that result from the essential properties of the ladder operators $\hat{a}, \hat{a}^\dagger$ and their commutation relations.

Anyway, where have we seen that?

$$\frac{\partial}{\partial t}\hat{a} = -i\omega\hat{a}$$

$$\hat{a} = e^{-i\omega t}$$

Plucked right out of

$$\psi(t) = e^{-i\omega t}\psi(0) = e^{-iEt/\hbar}\,\psi(0)$$

It's the phase operator that tells us how the wavefunction (state) evolves over time! Trumpets! Fanfare! $\hat{a}$ is the operator that gives a state an energy kick up to the next energy state on the ladder. There's the connection between state space and wave mechanics.

———————————————————————

Whew! I'm bogged down in the harmonic oscillator. Bogged down with the abstractions. What are the number operators? What's $|E\rangle$? What's it mean $|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle$? What's the relation of $\psi(x)$ to all the brackets? Maybe back to the big picture to figure out all the parts.

Take a look at that parabolic quantum well.

Quantum harmonic oscillator
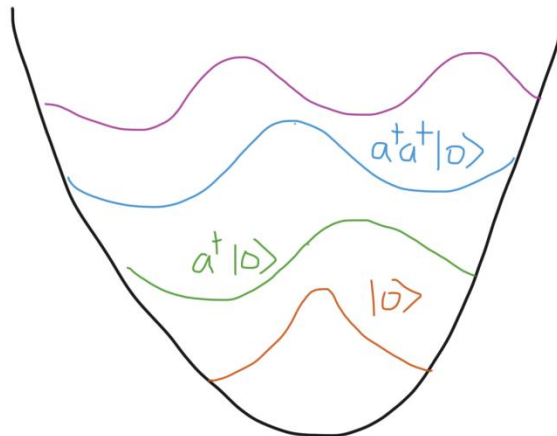
There's the potential, built into the geometry of the well. There are the number operators, counting rungs up the ladder of the wavefunctions. There's the lowest energy level, down at the bottom. There's the momentum built into the wavenumber at each energy level.

Focus on them counting operators. What information do they contain? Knowing the energy level (number) should provide enough information to fill in all the details. x, the only variable, is determined by the parabolic geometry. So the wavefunction, fitting waves into the parabola, is determined, and momentum is also determined (by the $\frac{\partial}{\partial x}$ operator). All you need to know is the energy level (number operator) and the potential.

Good! See if we can make it work!

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We've got a number state $|E\rangle$. What's the corresponding wavefunction?

Figure out the expectation value $\langle E|E\rangle$. No help there, just a number. But here's what we can do. We know that $\langle \hat{a}^\dagger E|\hat{a}^\dagger E\rangle = (N+1)\hbar\omega \langle E|E\rangle$. If we just had a bottom line, the lowest energy level, we could bootstrap up to all the others. Well, we've got that.

$$E_0 = \left(N_0 + \frac{1}{2}\right)\hbar\omega$$

$N_0 = 0$ by definition at the lowest energy state. So the ground state energy is $\frac{1}{2}\hbar\omega$ as per we-already-know-that.

Now find the wavefunction at that lowest state from

$$\hat{a}|\psi_0\rangle = \sqrt{\frac{m\omega}{2\hbar}}\left\langle\psi_0\left|\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)\right|\psi_0\right\rangle = 0$$

The lowering operator acting on the lowest energy state gives you no state at all.

Solve the differential equation and you're set. You've got the ground level energy and you can construct the ladder up from there.

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Here's another perspective from R. Shankar's text. Shankar dives into the weeds, but he's careful to emphasize the essentials. Here are the essentials.

$\hat{a}$ and $\hat{a}^\dagger$ are defined from the Hamiltonian. From that, note $[\hat{a}, \hat{a}^\dagger] = 1$, $[\hat{H}, \hat{a}^\dagger] = \hat{a}^\dagger$ and $[\hat{H}, \hat{a}] = -\hat{a}$. That follows from the algebra of those operators. Note that the coefficients of the raising and lowering operators convert to units of energy.

Most important of the operators for building the ladder:

$$H = \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)\hbar\omega = \left(n + \frac{1}{2}\right)\hbar\omega$$

where n is Shankar's number operator. It records which rung on the energy ladder we're at. Makes sense, $\hat{a}^\dagger a \equiv n$. Operate on an energy eigenstate with a takes you down a rung; then operate with $\hat{a}^\dagger$ and you're back where you started, on the same energy level. Now climb the ladder. Energy of the ground state $|0\rangle$ is

$$\left(n + \frac{1}{2}\right)\hbar\omega = \left(0 + \frac{1}{2}\right)\hbar\omega = \frac{1}{2}\hbar\omega$$

Energy of the second energy level

$$\left(1 + \frac{1}{2}\right)\hbar\omega = \frac{3}{2}\hbar\omega$$

and so on.

Next abstraction, rescale the energy into quantum units of $\hbar\omega$. Operator $\hat{H} \equiv \frac{H}{\hbar\omega}$ and eigenvalues $\varepsilon \equiv \frac{E}{\hbar\omega}$. Those are integer numbers. They scale the energy ladder of the harmonic oscillator into unit steps of $\hbar\omega$. We can label the rungs of the energy ladder by those numbers. Call them $|n\rangle$, energy states in the harmonic oscillator labeled by which rung of the ladder. That's all we need to know to get the rest of the information about the state, e.g. it's particular energy. That's all we need to know to figure out the spectrum.

At which we should take a moment to review Shankar's take-aways from this study of the SHO in a parabolic potential.

1. Energy is quantized.
2. There are unit steps $\hbar\omega$ between energy levels.

3. Ground state energy is $\frac{\hbar\omega}{2}$, not zero.
4. Wavefunctions leak out beyond the potential well, as per Adams' discussion of exponential decay at the boundaries.
5. Wavefunctions are either symmetric or antisymmetric in the potential well, i.e. cosine-like wavepackets or sine-like.
6. Because of 5. above, the probability distribution for locating a particle does not follow the classical predictions. (Compare graphs of $|n_0\rangle$ and $|n_1\rangle$. Particle most probably at the center of the well in the ground state but probability zero to be found at the center in $|n_1\rangle$.

Quantum harmonic oscillator
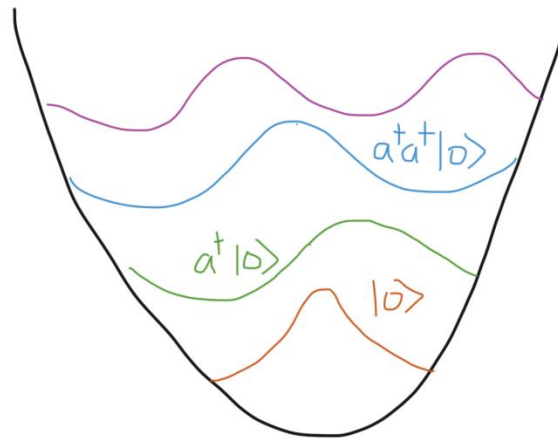
Onward to the matrix representation of energy eigenstates in Hilbert space.

Consider the Hamilton operator acting on a number state.

$$\hat{H}a|\varepsilon\rangle = (\epsilon - 1)a|\varepsilon\rangle$$

Work it out.

$$\hat{H}a|\varepsilon\rangle = \left(a\hat{H} + [\hat{H}, a]\right)|\varepsilon\rangle = (a\varepsilon - a)|\varepsilon\rangle = (\varepsilon - 1)a|\varepsilon\rangle$$

as advertised. We've substituted the energy eigenvalue ε for the operator in the last steps.

Take a close look. What that's telling us is that $(\varepsilon - 1)$ is the eigenvalue for the operator $\hat{H}$ acting on the original state $a|\varepsilon\rangle$. Since the number operators identify the states, then $a|\varepsilon\rangle$ itself must be the same as the state $|\varepsilon - 1\rangle$ up to a phase coefficient. Since the number operator

tells us everything we need to know about the energy state, if an operator transforms the original state $a|\varepsilon\rangle$ into state $(\varepsilon - 1)a|\varepsilon\rangle$ then $a|\varepsilon\rangle$ must equal $|\varepsilon - 1\rangle$ up to a phase factor.
$\widehat{H}a|\varepsilon\rangle = (\varepsilon - 1)a|\varepsilon\rangle = \widehat{H}|\varepsilon - 1\rangle = (\varepsilon - 1)|\varepsilon - 1\rangle$ up to a phase factor.

$$a|\varepsilon\rangle = C_n|\varepsilon - 1\rangle$$

Similarly we can show that

$$\widehat{H}\hat{a}^\dagger|\varepsilon\rangle = (\epsilon + 1)\hat{a}^\dagger|\varepsilon\rangle = (\epsilon + 1)C_{n+1}|\varepsilon + 1\rangle$$

The lowering and raising operators take us down and up the energy ladder. Up is okay. We can go forever up. Down must have a floor. A minimum ground state energy. What is it? Here's one advantage of Shankar's approach.

 At the ground state

$$a|\varepsilon_0\rangle = 0$$

Note the zero here is not a state, not an eigenvalue. It's a representation of nada. Such a condition, a state below the ground state, does not exist.

Well, if you act on a non-existing state with the raising operator, you still get nada.

$$\hat{a}^\dagger a|\varepsilon_0\rangle = 0$$

Now apply the operator relations.

$$\hat{a}^\dagger a|\varepsilon_0\rangle = \left(\widehat{H} - \frac{1}{2}\right)|\varepsilon_0\rangle = \left(\varepsilon_0 - \frac{1}{2}\right)|\varepsilon_0\rangle = 0$$

So

$$\varepsilon_0 = \frac{1}{2}$$

or, translated back into units of $\hbar\omega$

$$E_0 = \frac{\hbar\omega}{2}$$

Ground state energy is $\frac{\hbar\omega}{2}$. That's how we figure it out. Back and forth between number operators and the physical Hamiltonian.

Now let's go after those coefficients to the ladder operators. Here things get a little confusing with labels. Shankar (and Zwiebach, also) seem to be using the number labels and the energy level labels interchangeably. They both represent which step we are on the (quantized) energy spectrum. Anyway, using the number labels now:

Given

$$a|n\rangle = C_n|n - 1\rangle$$

it follows

$$\langle n|\hat{a}^\dagger a|n\rangle = \langle n - 1|C_n{}^\dagger C_n|n - 1\rangle$$

By previous definition, $\hat{a}^\dagger a \equiv n$, the number operator. Rewrite.

$$\langle n|n|n\rangle = \langle n - 1|C_n{}^\dagger C_n|n - 1\rangle$$

So

$$C_n{}^\dagger C_n = C_n{}^2 = n$$

$$C_n = \sqrt{n}$$

Coefficient for the eigenstate generated by the lowering operator is $\sqrt{n}$, the square root of the energy level of the original state. Similarly, coefficient for the eigenstate after the raising operator is $\sqrt{n + 1}$.

$$\hat{a}^\dagger|n\rangle = \sqrt{n + 1}\ |n + 1\rangle$$

And now – drum roll! – we can put it all together in operator matrix and normalized eigenstate vector form. Take a look.

$$a = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 \\ \vdots & 0 & 0 & 0 & \sqrt{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$a^\dagger = \begin{bmatrix} 0 & 0 & 0 & \cdots & & \\ 1 & 0 & 0 & 0 & & \\ 0 & \sqrt{2} & 0 & 0 & & \\ \vdots & 0 & \sqrt{3} & 0 & 0 & \\ & 0 & 0 & \sqrt{4} & 0 & 0 \\ & & & 0 & \ddots & 0 \end{bmatrix}$$

$$E = \hbar\omega \begin{bmatrix} \frac{1}{2} & 0 & & \cdots & \\ 0 & \frac{3}{2} & & & \\ & & \frac{5}{2} & & \\ \vdots & & & \frac{7}{2} & \\ & & & & \frac{9}{2} \\ & & & & & \ddots \end{bmatrix}$$

So when you raise the $n = \varepsilon = 3$ eigenstate, for example

$$\hat{H}a^\dagger|3\rangle = \hbar\omega \begin{bmatrix} \frac{1}{2} & 0 & & \cdots & \\ 0 & \frac{3}{2} & & & \\ & & \frac{5}{2} & & \\ \vdots & & & \frac{7}{2} & \\ & & & & \frac{9}{2} \\ & & & & & \ddots \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \cdots & & \\ 1 & 0 & 0 & & \\ 0 & \sqrt{2} & 0 & & \\ \vdots & & \sqrt{3} & 0 & \\ & & & \sqrt{4} & 0 \\ & & & & \ddots & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

$$= \hbar\omega \begin{bmatrix} \frac{1}{2} & 0 & & & & \cdots \\ 0 & \frac{3}{2} & & & & \\ & & \frac{5}{2} & & & \\ \vdots & & & \frac{7}{2} & & \\ & & & & \frac{9}{2} & \\ & & & & & \ddots \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sqrt{3} \\ 0 \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ \left(\frac{7}{2}\right)\sqrt{3} \\ 0 \\ \vdots \end{bmatrix} \hbar\omega$$

Pretty slick! Take a close look at how those row and column indices are working to raise from energy level 3 to 4.

But that's not quite the whole story. We've got to construct the spectrum from the ground state up, rung by rung up the energy ladder. And we have to normalize. Start with orthonormal eigenstates and you have to maintain orthonormal states. Not too bad, really. Here's the grand finale.

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$$

Start with the ground state. Operate n times with the raising operator, as per the example above. Divide each time by the coefficient to maintain unit norm. Ta da! You've got the spectrum of the quantum harmonic oscillator!

Best introduction to next idea is email I sent to Prof. Zwiebach re: his derivation of the Schrodinger equation from the time operator, U:

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I am working through your MIT OCW Physics 8.05. In Lecture 12, Dynamics, you derive the Schrodinger equation from the unitary time operator. It struck me that the result is (or sure looks like) a continuity equation, where the Hamiltonian operator is a 'current' and the norm of the state vector is a conserved 'charge.'

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \frac{\partial U_0^t}{\partial t}U_0^{t\dagger}|\psi(t)\rangle$$

A continuity relation seems to make sense, a la Noether, with regard to conservation of energy. Anyway, I was curious if this notion has any merit. On superficial review of the literature, I find SE as continuity of probability, but I don't see any reference to this way of thinking about continuity of the time-dependent SE.

I'm an old geezer, retired high school teacher, trying to figure out quantum gravity. I've been brushing up on QM. I've sure enjoyed your video lectures, and I've learned a whole lot. Thanks very much for making these ideas accessible to the rest of us.

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Here's the full monty. Start with the Bloch sphere.

Pole states:

$$|i+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$|i-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

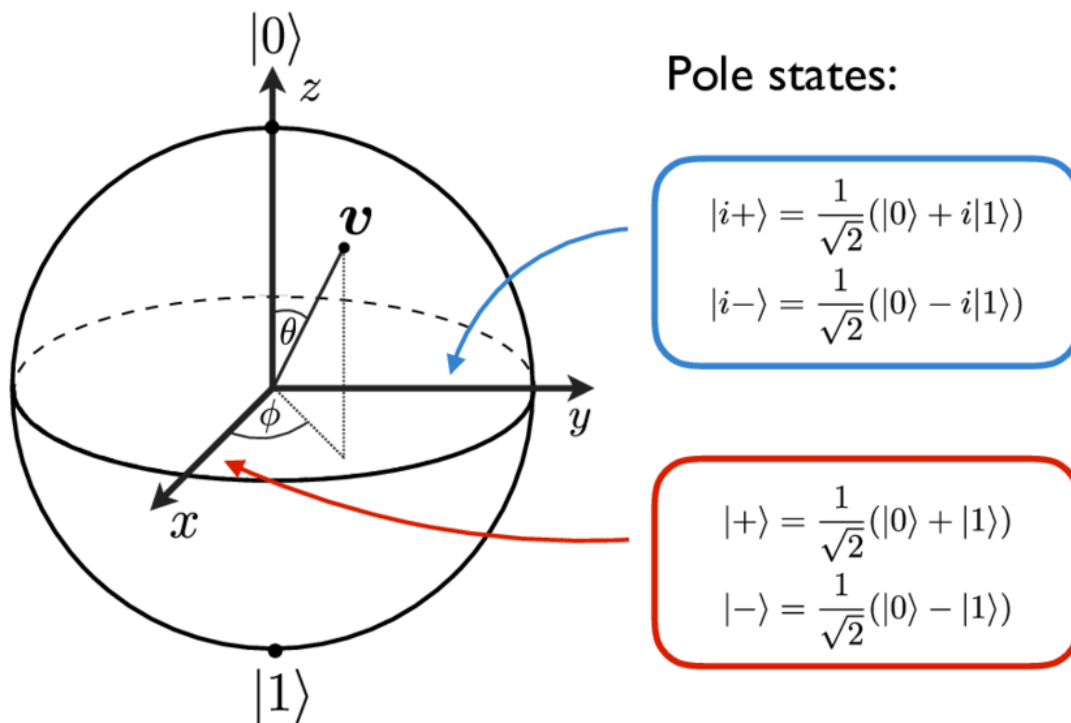$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Figure. Bloch sphere. Credit Andreas Ketterer. 2016. Modular variables in quantum information. Thesis.

A unitary time operator rotates the (normalized) state vector around the Bloch sphere in Hilbert space time step by step. Let U_0^t represent the operator that takes the state $|\psi(0)\rangle$ to $|\psi(t)\rangle$.

$$U_0^t |\psi(0)\rangle = |\psi(t)\rangle$$
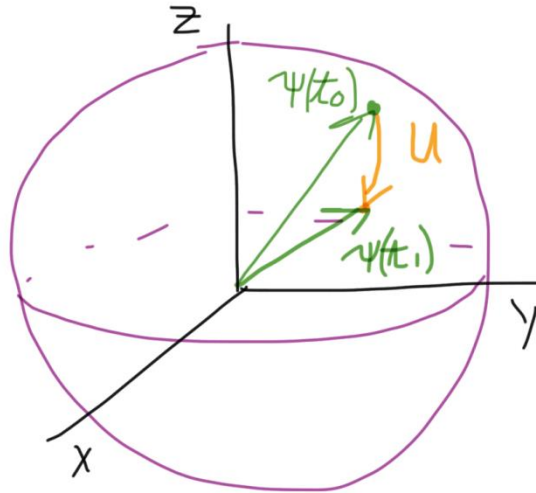
Time operator on the Bloch sphere

Figure. Time operator on the Bloch sphere. U updates the wavefunction in time increments.

The unitary time operator is unique. It evolves <u>any</u> state to its next time increment. And by definition it's reversible

$$U_0^0 = I$$

and

$$U_t^0 U_0^t = I$$

so

$$U_0^t = U_t^{0\dagger}$$

I'm using the indices here for general purposes. They could be any $t1$ and $t2$.

Also

$$U_{t2}^{t3} U_{t1}^{t2} = U_{t1}^{t3}$$

With that, we're all set to derive the time-dependent Schrodinger equation. Start with

$$\frac{\partial}{\partial t} |\psi(t)\rangle = \frac{\partial}{\partial t} (U_0^t |\psi(0)\rangle)$$

The only time dependence on the rhs is in the unitary operator, so

$$\frac{\partial}{\partial t}|\psi(t)\rangle = \frac{\partial U_0^t}{\partial t}|\psi(0)\rangle$$

We want the same eigenstate $|\psi(t)\rangle$ on both sides.

$$\frac{\partial}{\partial t}|\psi(t)\rangle = \frac{\partial U_0^t}{\partial t}U_t^0|\psi(t)\rangle = \frac{\partial U_0^t}{\partial t}U_0^{t\dagger}|\psi(t)\rangle$$

Call that operator on the rhs Λ.

$$\Lambda \equiv \frac{\partial U_0^t}{\partial t}U_0^{t\dagger}$$

and

$$\Lambda^\dagger \equiv U_0^t\frac{\partial U_0^{t\dagger}}{\partial t}$$

Note that the lambda operators are currents! Just like

$$\frac{\partial\,\psi(x)}{\partial t}\psi^*(x)$$

is the probability current.

Claim is that the lambda operators anti-commute.

$$\Lambda + \Lambda^\dagger = 0$$

Easy to show:

$$\frac{\partial}{\partial t}\left(U_0^t U_0^{t\dagger}\right) = \frac{\partial}{\partial t}I = \frac{\partial U_0^t}{\partial t}U_0^{t\dagger} + U_0^t\frac{\partial U_0^{t\dagger}}{\partial t} = \Lambda + \Lambda^\dagger = 0$$

Multiply both sides by a factor $i\hbar$. That converts the lambdas to commuting operators. Then we're set.

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = i\hbar\frac{\partial U_0^t}{\partial t}U_0^{t\dagger}|\psi(t)\rangle = i\hbar\,\Lambda|\psi(t)\rangle$$

Now $i\hbar\,\Lambda$ is a unitary, time-step Hermitian operator. What's in a name? Call it $\hat{H}$.

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

Schrodinger! And cool thing is, by all appearances it's a continuity equation. Λ hence $\hat{H}$ is a current. It's a current in time. It's the flow of time.

$$\hat{H} = i\hbar \frac{\partial U_0^t}{\partial t} U_0^{t\dagger}$$

That derivative-times-operator is a current just like the probability density current $J = \frac{\partial \psi(x)}{\partial x} \psi(x)^*$. The conserved 'charge' is the norm of the state vector.

$$\langle \psi(t)| \, i\hbar \frac{\partial}{\partial t} \, |\psi(t)\rangle - \hat{H} = 0$$

A charge. A Noether current. Symmetry. It's all right there. Energy is conserved.

Well, hardly surprising. We've seen that from the commutation relations and a bunch of other things before. Still, pretty cool that it appears in Schrodinger. Or that you can derive Schrodinger from Noether.

Lecture 13: Dynamics (cont'd) and the Heisenberg operator

What's next is to understand the unitary time evolution operator in terms of the Hamilton operator. Any calculation in quantum mechanics generally starts with a Hamiltonian. That's the physics. From that we want to figure out U, to help with the maths.

Finding the relation is straightforward. Return to Schrodinger.

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = i\hbar \frac{\partial}{\partial t} (U_0^t |\psi(0)\rangle) = \hat{H}|\psi(t)\rangle = \hat{H} \, (U_0^t |\psi(0)\rangle)$$

Since the only time-dependence is in the $U_0^t{}'s$ we can write

$$i\hbar \frac{\partial}{\partial t} U_0^t = \hat{H} \, U_0^t$$

and solve! For a time-dependent Hamiltonian,

$$U_0^t = e^{-iHt/\hbar}$$

Similarly for a 'slightly' time-dependent Hamiltonian.

$$U_0^t = e^{-iH(t_1-t_0)/\hbar}$$

The whole-shebang Hamiltonian-time-dependence follows:

$$U_0^t = exp\left(-\frac{i}{\hbar}\int_0^t H(t)\,dt\right)$$

That comes from the Taylor series of the exponential with the time ordering operator. See Prof. Z's notes for details. Anyway, makes sense just by the looks of it. Time operator increments step by step over time, driven by the Hamiltonian.

Onward to Heisenberg operators. By our previous definition,

$$A_H \equiv U_0^{t\dagger} A_S U_0^t$$

where the H and the S subscripts refer to Heisenberg and Schrodinger. Schrodinger operators are all the usuals, $\hat{x}, \hat{p}, \hat{H}$, etc. The dynamics now is shifted to the Heisenberg operator.

By the definition,

$$\langle \psi(t)|A_s|\psi(t)\rangle = \left\langle \psi(0)\left|U_0^{t\dagger} A_S U_0^t\right|\psi(0)\right\rangle = \langle \psi(0)|A_H|\psi(0)\rangle$$

That's handy. The Heisenberg operator allows us to choose an initial state to study the dynamics, and we can stick with that state through our calculations.

This definition gives a bunch of handy relations linking Heisenberg to the usual Schrodinger operators.

$$C_S = A_S B_S \;\rightarrow\; C_H = U_0^{t\dagger} A_S U_0^t U_0^{t\dagger} B_S U_0^t = U_0^{t\dagger} A_S I B_S U_0^t = U_0^{t\dagger} A_S B_S U_0^t = C_H$$

and the commutator relations are unchanged.

$$[A_H, B_H] = \left[U_0^{t\dagger} A_S U_0^t, U_0^{t\dagger} B_S U_0^t\right] = \left(U_0^{t\dagger} A_S U_0^t\right)\left(U_0^{t\dagger} B_S U_0^t\right) - \left(U_0^{t\dagger} B_S U_0^t\right)\left(U_0^{t\dagger} A_S U_0^t\right)$$

$$= \left(U_0^{t\dagger} A_S B_S U_0^t\right) - \left(U_0^{t\dagger} B_S A_S U_0^t\right) = A_H B_H - B_H A_H = [A_H, B_H]$$

sure enough! Note that out of laziness I've dropped the operator hat symbols.

From this we can prove anew that $i\hbar\frac{\partial}{\partial t}A_H = [A_H, H_H]$.

Start with Schrodinger.

$$i\hbar\frac{\partial}{\partial t}A_H = i\hbar\frac{\partial}{\partial t}U_0^{t\dagger} A_S U_0^t = i\hbar\left(\frac{\partial U_0^{t\dagger}}{\partial t}A_S U_0^t + U_0^{t\dagger}\frac{\partial A_S}{\partial t}U_0^t + U_0^{t\dagger} A_S\frac{\partial U_0^t}{\partial t}\right)$$

Convert the time derivatives of the unitary operators to their Hamiltonians.

$$= -U_0^{t\dagger} \hat{H}_S A_S \, U_0^t + i\hbar \left(U_0^{t\dagger} \frac{\partial A_S}{\partial t} U_0^t \right) + U_0^{t\dagger} A_S \hat{H}_S \, U_0^t$$

$$= -U_0^{t\dagger} \hat{H}_S A_S \, U_0^t + U_0^{t\dagger} A_S \hat{H}_S \, U_0^t + i\hbar \left(U_0^{t\dagger} \frac{\partial A_S}{\partial t} U_0^t \right)$$

$$= [A_H, H_H] + i\hbar \left(U_0^{t\dagger} \frac{\partial A_S}{\partial t} U_0^t \right)$$

Now if A_S , the Schrodinger operator, has no time dependence, the last term disappears, and we're left with the commutator relation as proved.

None of this is real surprising, but it's interesting to see how those Heisenberg operators work.

Take a look now how the Heisenberg picture simplifies our understanding of the physics. Following comes from Prof. Z as well as Shankar's text.

First note the geometric relation between the Schrodinger and Heisenberg pictures. Schrodinger's (vector) states rotate around Hilbert space against fixed coordinates (the eigenstates). Heisenberg, on the other hand, says the states are fixed and the coordinates rotate. It's the basis vectors, as represented in the operators, that are changing over time. The physics in both pictures is the same. Calculations come out the same. Just a different way of looking at the world, and Shankar says there's a whole lot of other models we might build. Lots of room for creative thinking.
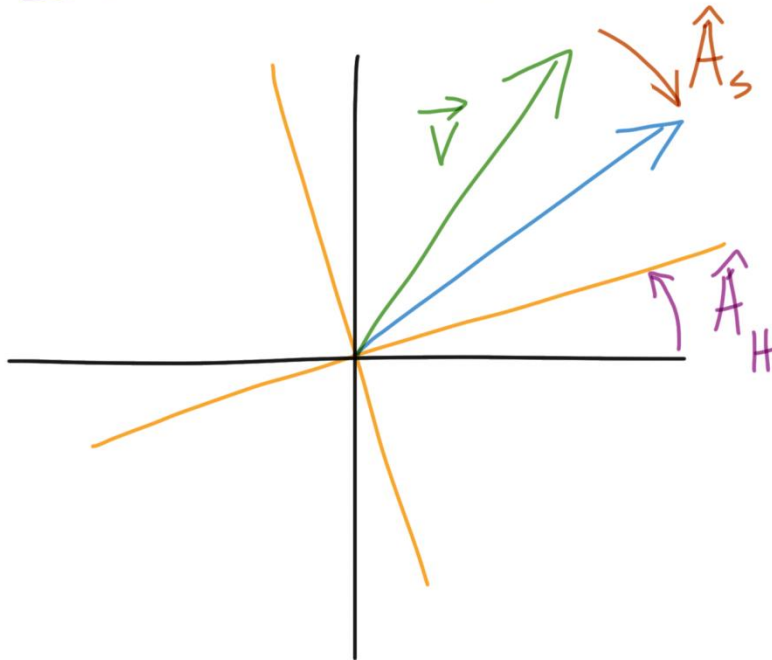
Figure. Schrodinger vs. Heisenberg operators. Schrodinger rotates the wavefunction, Heisenberg rotates the axes.

The Heisenberg picture simplifies our thinking about the physics. H makes the dynamics look the same as in classical physics. For example, in the quantum harmonic oscillator

$$\frac{\partial \hat{x}}{\partial t} = [\hat{x}, \hat{H}] = \left[\hat{x}, \left(-\frac{i}{\hbar} \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 \right) \right] = \left[\hat{x}, \left(-\frac{i}{\hbar} \frac{\hat{p}^2}{2m} \right) \right]$$

Whoa! What happened to the second term in the Hamiltonian? Well, it commutes with $\hat{x}$ so disappears from the equation.

$$\frac{\partial \hat{x}}{\partial t} = -\frac{i}{\hbar} \frac{1}{2m} [\hat{x}, \hat{p}^2] = -\frac{i}{\hbar} \frac{1}{2m} ([\hat{x}, \hat{p}]\hat{p} + [\hat{x}, \hat{p}]\hat{p})$$

where that last step pulled out one of the two $\hat{p}$ operators for each term; you have to calculate the commutator twice; once for each $\hat{p}$. So

$$\frac{\partial \hat{x}}{\partial t} = -\frac{i}{\hbar} \frac{1}{2m} ([\hat{x}, \hat{p}]\hat{p} + [\hat{x}, \hat{p}]\hat{p}) = -\frac{i}{\hbar} \frac{1}{2m} (2i\hbar\hat{p}) = \frac{\hat{p}}{m}$$

Just as in classical mechanics! Heisenberg operators return the classical equation of motion!

Same for $\frac{\partial \hat{p}}{\partial t}$.

$$\frac{\partial \hat{p}}{\partial t} = [\hat{p}, \hat{H}] = \left[\hat{p}, \left(-\frac{i}{\hbar}\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right)\right] = \left[\hat{p}, \left(\frac{1}{2}m\omega^2\hat{x}^2\right)\right]$$

Same rationale here for dropping the first term in the Hamiltonian. It commutes with $\hat{p}$.

$$\frac{\partial \hat{p}}{\partial t} = \frac{i}{\hbar}\frac{1}{2}m\omega^2[\hat{p}, \hat{x}^2] = \frac{i}{\hbar}\frac{1}{2}m\omega^2([\hat{p}, \hat{x}]\hat{x} + [\hat{p}, \hat{x}]\hat{x})$$

$$\frac{\partial \hat{p}}{\partial t} = \frac{i}{\hbar}\frac{1}{2}m\omega^2(2i\hbar\hat{x}) = -m\omega^2\hat{x}$$

Shankar's other observation deserves repeat. Back to the definition. Because the Heisenberg operators do all the lifting for time evolution we can solve the dynamics just based on some initial state, which presumably we can determine. Which is the whole point. We <u>know</u> a state to start with and we want to see how it evolves.

Lecture 14: Coherent states

Coherent states are replicates. Well, sort of. They share the same energy but differ in other observables.

Take the quantum harmonic oscillator for example. Lowest eigenstate, the ground state, has energy $\frac{1}{2}\hbar\omega$. But we could shift the apparatus a bit and the ground state over there is the same $\frac{1}{2}\hbar\omega$ as the ground state here. Coherent.

Prof. Zwiebach starts the lecture with a review of dynamics: position and momentum operators as functions of time.

$$\hat{x}(t) = \hat{x}(0)cos(\omega t) + \frac{\hat{p}}{m\omega}(0)sin(\omega t)$$

$$\hat{p}(t) = \hat{p}(0)cos(\omega t) - m\omega\hat{x}(0)sin(\omega t)$$

New addition is the Heisenberg dynamics of the ladder operators.

$$\hat{a}_H = e^{-i\omega t}\hat{a}$$

and

$$\hat{a}_H{}^\dagger = e^{i\omega t}\hat{a}^\dagger$$

Essential tools for understanding coherent states are the translation operators.

$$T_{x_0} \equiv e^{-i\hat{p}x_0/\hbar}$$

What it does is increment the position by an interval x_0 . Maybe a more consistent symbolic representation would be

$$T_x^{x_0} \equiv e^{-i\hat{p}x_0/\hbar}$$

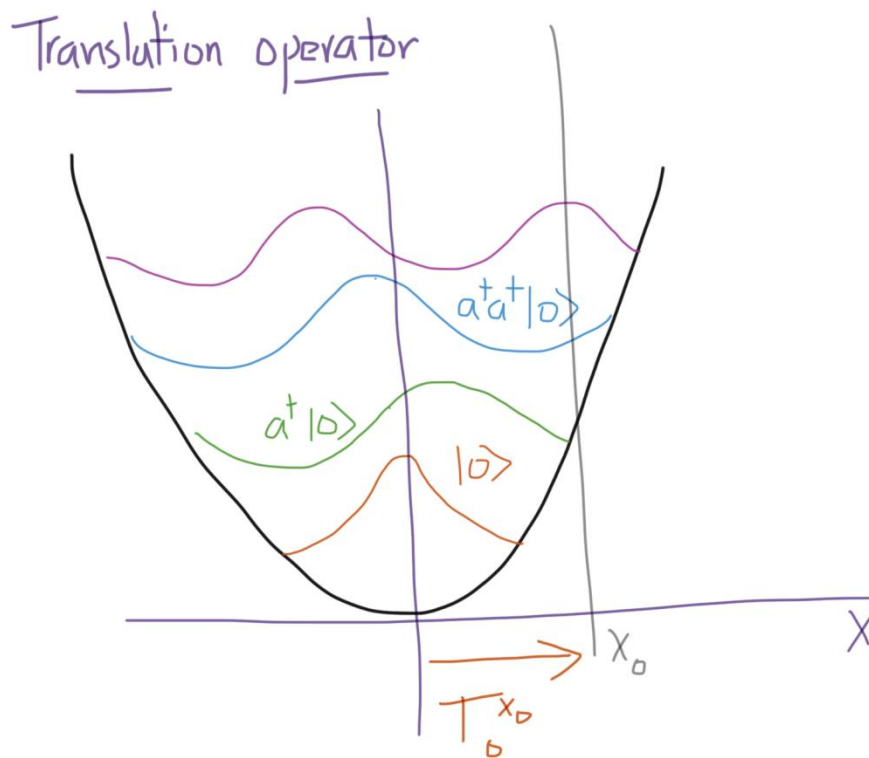That is, the translation operator takes the state from position x to position $x + x_0$.

Figure. Heisenberg translation operator on quantum harmonic oscillator.

Note the relation to the momentum operator. $\hat{p} = \frac{i}{\hbar}\frac{\partial}{\partial x}$. Translation by delta-x along the quantum oscillator boosts the momentum up the ladder. Makes me wonder if the whole universe is a quantum harmonic oscillator . . . That seems consistent with the differential representation of the momentum operator anyway.

Note the parallel to the unitary time operator. U_0^t takes the state from time 0 , some time we choose to call zero on our stopwatch, to some later time t . U_0^t is moving the state through time.

$T_x^{x_0}$ is moving the state through space. Which raises the question, is there some relativistic relation $T_x^{x_0\,2} - U_0^{t\,2} = S^2$? A metric?

Anyway, back to the standard stuff. Given the symbolic representation, the algebra of the translation operator is pretty clear from the physics.

$$T_{x_0}{}^\dagger = T_{-x_0} = \left(T_{x_0}\right)^{-1}$$

so

$$T_{x_0}{}^\dagger T_{x_0} = I$$

That is, you've translated to a new position then right back to where you started.

$$T_{x_0} T_{y_0} = T_{x_0 + y_0}$$

That is, translation steps are additive.

Note the representation for T_{x_0} acting on the position operator.

$$T_{x_0}{}^\dagger \hat{x} \, T_{x_0} = \hat{x} + x_0 I$$

Think about that one. Talking operators here. lhs is an operator. It's rotating basis axes relative to a state vector.

$$\langle \psi(x) | T_{x_0}{}^\dagger \hat{x} \, T_{x_0} | \psi(x) \rangle = \langle \psi(x) | \hat{x} \, | \psi(x) \rangle + x_0 I$$

Take a look at the state vectors. (See Figure above, translation on the quantum harmonic oscillator.)

$$T_{x_0} | x \rangle = | x + x_0 \rangle$$

Switching between state vectors and the wavefunction

$$| \psi \rangle \; \to \; \psi(x)$$

$$T_{x_0} | \psi \rangle \; \to \; \psi(x - x_0)$$

Note the minus sign. That's the usual rule for translating functions across the coordinates. Minus sign if you move the function to the right.

Now that we've got the tools, on to coherent states. In the quantum harmonic oscillator

$$| \tilde{x}_0 \rangle \equiv T_{x_0} | 0 \rangle = e^{-ipx/\hbar} | 0 \rangle$$

That's it. By definition, a coherent state is the ground state of the harmonic oscillator translated is position but still in the ground state. Slide the pendulum a bit to the right. Move the snowboard pipe a tad further around the hill. Energy states are unchanged. Coherent.

Here's an aside. How do you run a unit analysis quickly so you can understand, for example, the coefficients in the expectation values? It crossed my mind a good start is in the equivalents for the Planck constant.

$$\hbar = Et = px$$

Play around a bit, makes sense.

$$E = p\frac{x}{t}$$

standard units for kinetic energy, momentum times velocity.

Or

$$\frac{E}{x} = \frac{p}{t}$$

Hamilton equations of motion.

Take a look then at the position and momentum equations of motion we've derived in our dynamics.

$$\hat{x}(t) = \hat{x}(0)cos(\omega t) + \frac{\hat{p}}{m\omega}(0)sin(\omega t)$$

That coefficient $\frac{\hbar}{m\omega}\hat{p}(0)$ should give units of x . Let's see. First term on the rhs is fine. Second term needs some reckoning. Units.

$$\frac{1}{m\omega}\hat{p}(0) = \frac{p}{E/_{\omega x^2}} = \frac{p}{\hbar/_{x^2}} = \frac{p}{px/_{x^2}} = x$$

where I simplified at the second step using the harmonic oscillator $V = \frac{1}{2}m\omega^2 x^2$ and, third step, the units $\omega = \frac{1}{t}$.

It works! No great surprise, but maybe it will help keep track of the coefficients.

Back to Zwiebach and the coherent states. A few more key ingredients.

$$\langle \tilde{x}_0 | \tilde{x}_0 \rangle = 1$$

General expectation values under translations:

$$\langle \tilde{x}_0 | \hat{A} | \tilde{x}_0 \rangle = \langle 0 | T_{x_0}{}^\dagger \hat{A} T_{x_0} | 0 \rangle$$

So, for example, expectation value for position under translation isn't surprising:

$$\langle \tilde{x}_0 | \hat{x} | \tilde{x}_0 \rangle = \langle 0 | (x + x_0) | 0 \rangle = x_0$$

But expectation value for momentum is a bit counter-intuitive:

$$\langle \tilde{x}_0 | \hat{p} | \tilde{x}_0 \rangle = 0$$

That's because it's a (Schrodinger) stationary state, wavefunction just sitting there. No momentum. Finally, expectation value for energy

$$\langle \tilde{x}_0 | \hat{H} | \tilde{x}_0 \rangle = \langle 0 | \hat{H} | 0 \rangle + \frac{1}{2} m \omega^2 x_0{}^2 = \frac{1}{2} \hbar \omega + \frac{1}{2} m \omega^2 x_0{}^2$$

Energy in the coherent state is augmented by the potential at the (displaced) position x_0. Makes sense.

And for future reference:

$$\langle \tilde{x}_0 | \hat{x}^2 | \tilde{x}_0 \rangle = \tilde{x}_0{}^2 + \frac{\hbar}{2m\omega}$$

$$\langle \tilde{x}_0 | \hat{p}^2 | \tilde{x}_0 \rangle = \frac{m\omega\hbar}{2}$$

$$\langle \tilde{x}_0 | \hat{x}\hat{p} + \hat{p}\hat{x} | \tilde{x}_0 \rangle = 0$$

Onward to the dynamics. We'll use the good ol' Heisenberg operators so we can access classical thinking.

$$\langle \tilde{x}_0(t) | \hat{A}_S | \tilde{x}_0(t) \rangle = \langle \tilde{x}_0 | \hat{A}_H | \tilde{x}_0 \rangle$$

Try it out on the position operator. See what happens to the coherent state position over time.

$$\langle \tilde{x}_0 | \hat{x}_H | \tilde{x}_0 \rangle = \left\langle \tilde{x}_0 \left| \hat{x}(0)cos(\omega t) + \frac{\hat{p}(0)}{m\omega} sin(\omega t) \right| \tilde{x}_0 \right\rangle = x_0 cos(\omega t)$$

As per the general results above, the momentum disappears. As it should. And looky! It's the good old classical equation! The position oscillates around zero.

Lecture 15: Coherent states and squeezed states

Prof. Zwiebach calculates the general coherent state, which includes the ladder operators, and he explains the squeezed state. The math is complicated, and I won't reproduce it here. Just lazy, I guess, but I'm getting antzy, want to head back to the frontier, quantum information and gravity. Time to get moving, finish up the QM review.

Squeezed states are worth some discussion, though. I've always wondered what they were. Prof. Z. explains them well. Here's the notion.

Take a coherent state in the ground state of a Hamiltonian. It has an uncertainty

$$\Delta x_1 = \sqrt{\frac{\hbar}{2m_1\omega_1}}$$

where the subscripts identify the particular Hamiltonian.

Now zap the system into a new Hamiltonian. Calculate the new uncertainty.

$$\Delta x_2 = \sqrt{\frac{\hbar}{2m_2\omega_2}} = \sqrt{\frac{m_1\omega_1}{m_2\omega_2}}\sqrt{\frac{\hbar}{2m_1\omega_1}}$$

Now if

$$\gamma \equiv \sqrt{\frac{m_1\omega_1}{m_2\omega_2}} < 1$$

as in if the energy of the second state is higher than the first, then the state has been squeezed. Think of a Gaussian. It isn't as wide as it was to start with. It has a sharper peak.
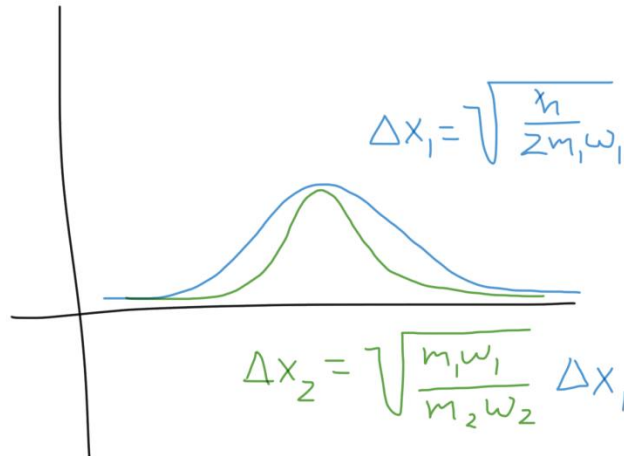
Figure. Squeezed state illustrated as Gaussian function squeezed by boost to higher potential.

Represent that transformation as an operator. If you want to build a squeezed state you need a squeezing operator.

$$S(\gamma) = e^{-\gamma/2(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a})}$$

Note that it is quadratic in the ladder operators, and note the order of those operators in the exponent. Annihilation op's have to be to the right, acting first on the state you're squeezing. Otherwise the whole thing blows up, driving the state upward with successive creation operators. Now define the squeezed vacuum state

$$|0_\gamma\rangle = S(\gamma)|0\rangle$$

Applications of squeezed states are really interesting. LIGO uses translation and squeezing operators to reduce noise in its detection system. The mirrors oscillate a bit because of thermal noise. That smears out the gravity wave signal. Solution: squeeze the detector photons so they're less exposed to mirror fluctuations and translate them to where they should be if the mirror was absolutely quiet.

$$|\alpha, \gamma\rangle = D(\alpha)S(\gamma)|0\rangle$$

Pretty cool! Squeezed states do marvelous things. Perform sharper measurements. Send sharper signals.

<u>Lecture 16</u>: Photon coherent states and two-state systems

Idea here is you can write the Hamiltonian for the electromagnetic field in a way that looks like the harmonic oscillator.

$$E = \frac{1}{2}(p^2 + \omega^2 q^2)$$

Beware; E here is the electromagnetic field. Looks the same as the harmonic oscillator but without mass. It's reasonable in units: $[pq] = [\hbar]$. So convert to operators and declare

$$H \equiv \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{q}^2)$$

where

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}} \, (\hat{a} + \hat{a}^\dagger)$$

and

$$\hat{p} = \sqrt{\frac{\hbar\omega}{2}} \, (\hat{a} - \hat{a}^\dagger)$$

so

$$H = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) = \hbar\omega \left(N + \frac{1}{2} \right)$$

N now is the photon energy. Just like ladder operator stuff in the harmonic oscillator. What's it all mean? Well consider the photon. It's a quantum of the electromagnetic field. Think of it as a coherent state, a mode in the stupendous harmonic oscillator well of the electromagnetic universe. It has an associated momentum and potential energy, oscillating as it is between its potential boundaries. Mass on a spring but no mass. Just the spring oscillating. That said, we can assign the usual ladder operators to the Hamiltonian just as in the QHO. Same maths.

All that said, we can think of the field itself as an operator.

$$\hat{E} = \varepsilon_0 \big(e^{-i\omega t} \hat{a} + e^{-i\omega t} \hat{a}^\dagger \big) sin(kz)$$

where the field is polarized along the z-axis.

<u>Lectures 17 and 18</u>: Two-state systems, ammonia and NMR

Two-state systems include e.g. spin states and the ammonia molecule: you can capture them neatly with a 2×2 Hamiltonian matrix thusly:

$$H = \begin{bmatrix} g_0 + g_3 & g_1 - g_2 \\ g_1 + g_2 & g_0 - g_3 \end{bmatrix} = g_0 I + g_1 \sigma_x + g_2 \sigma_y + g_3 \sigma_z$$

In this mathematical structure, all the dynamics involves some kind of 'precession.' Magnetic moment in a magnetic field as the prime example, of course, but same maths describe the ammonia molecule and other two-state systems.

Note the rubric to build models.
1. Find a likely Hamiltonian
2. Find the energy eigenstates and eigenvalues
3. Find the expectation values
4. Find the dynamics, i.e. the time evolution coefficients

Ammonia is really interesting. It's a two-state system, nitrogen either above or below the plane of hydrogen atoms, so it has an electric dipole. Put it in an electric field and you separate molecules by energy Δ above or below the ground state.
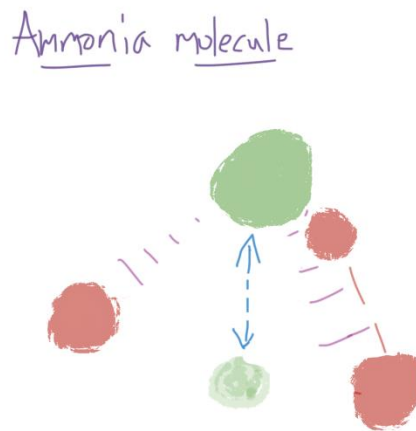

Ammonia molecule

Figure: ammonia molecule. Nitrogen (green) oscillates across the plane of the three hydrogens. It is a dipole molecule with characteristic flip frequency. An electric field separates the two states.

Eigenstates you can label, as per usual

$$|\uparrow\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } |\downarrow\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then the Hamiltonian becomes

$$H = \begin{bmatrix} \Delta & \varepsilon_0 E \\ \varepsilon_0 E & -\Delta \end{bmatrix}$$

From there you can calculate dynamics, which (no surprise) includes terms like $exp\left(\frac{i\omega t}{\hbar}\right)$ and $cos(\omega t)$, where ω is the Larmour (precession) frequency. From those dynamics you can calculate how long it takes, time T, for an up state to flip down. And that, my friend, lets you build masers!

Separate states with a gradient electric field. Send up state into a resonant cavity of just the right length such that the transit time $= T$. That's just right for the molecule to emit a photon of energy $= 2\Delta$. Photons pile up in the cavity. Let them leak out and you've got a maser. Nobel prize for Townes et al in 1964.
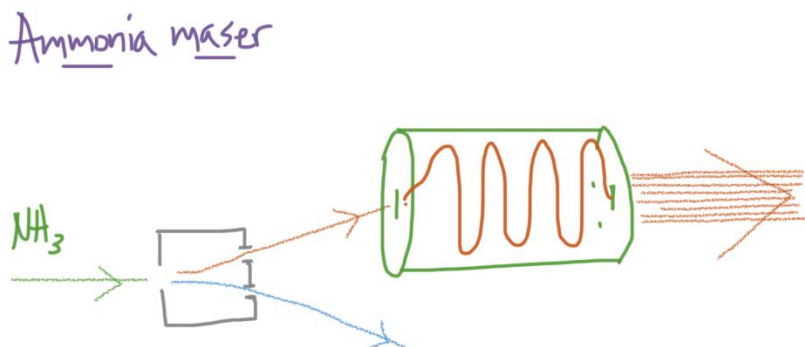

Figure: ammonia maser. An electric field splits ammonia beam into high- and low-energy states (relative to the field). High energy state enters resonant cavity with dimensions such that the beam drops to low energy and releases a photon as it traverses the cavity.

It's all right there in those matrix operators and state vectors and a little bit of math. (Well, quite a bit of math.)

NMR uses the same maths tools. Apparatus has a twist to it, though, a rotating magnetic field. Put your target nucleus in a really strong, constant B_z field. Add a rotating B field in the x-y plane. Nuclear spin precesses around B_z and also around the (rotating) B_x. Effect is to torque the spin axis down into the $x - y$ plane. As it spirals down, it radiates at the frequency of the rotating B_x. Tune the detectors to that frequency. You're seeing mostly the hydrogens in water water molecules. Strength of the signal depends on the water concentration and the composition of neighboring molecules. You can get even more information from the damping time and relaxation time; how long does it take to spiral down, and how long to revert to alignment along B_z.
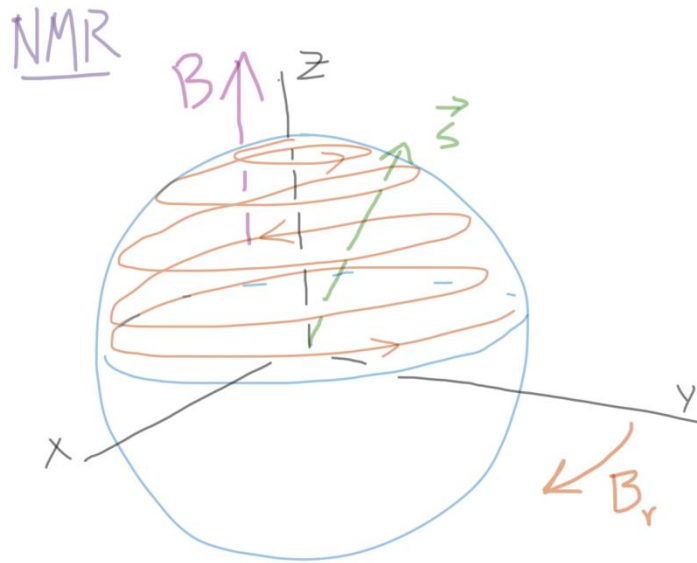
Figure NMR. In a constant external magnetic field B and rotating field in the $x\text{-}y$ plane B_r nuclear spin will precess from the z pole down into the $x\text{-}y$ plane. The process emits cyclotron radiation as it drops into the plane, and that information is used to construct an image.

<u>Lecture 19</u>: Tensor product and teleportation

I finally get it! Tensor product is not really a multiplication. It's a record-keeping system for multiparticle states.

Main idea is that you can't describe a multiparticle system just by listing the individual properties of all the component particles. It's not enough to know the position and momentum of each individual particle. Those particles are correlated. Their wavefunctions interact. You have to keep track of all those correlations, all that entanglement.

If V is the Hilbert space of one particle and W the Hilbert space of a second particle, then the Hilbert space of a system with both particles is $V \otimes W$. For example, given two spin-half particles, their vector spaces

$$V = W = \{ \, |+\rangle, |-\rangle \, \}$$

and

$$V \otimes W = \{ \, |+\rangle \otimes |+\rangle, |+\rangle \otimes |-\rangle, |-\rangle \otimes |+\rangle, |-\rangle \otimes |-\rangle \, \}$$

By convention we'll typically drop the $\otimes$ between state vectors.

$$V \otimes W = \{ \, |+\rangle|+\rangle, |+\rangle|-\rangle, |-\rangle|+\rangle, |-\rangle|-\rangle \, \}$$

Note that the dimension of the tensor product state is the product of the dimensions of the two component states. All the usual rules of linear algebra apply: scalar coefficients distribute and so do vector states.

$$au \otimes (v \otimes w) = a(u \otimes v) \otimes a(u \otimes w)$$

With those rules you can do all kinds of marvelous things. Like for example build a quantum teleportation system. The physical system in the following example uses spin states. Spin operators are unitary, so conserve probability and information. You implement the operators with varying magnetic fields. Them's what goes into the Hamiltonians we call quantum logic gates. Them's what's the physical instantiation of the operators. The gates are magnets.

Here's an illustration of spin operators on Bell states. We'll use them for teleportation. Define the Bell state

$$|\phi_0\rangle \equiv \frac{1}{\sqrt{2}}(|+\rangle|+\rangle + |-\rangle|-\rangle)$$

Unitary. Normalized. Perfect. Now operate with the spin operators. Note that we have to use augmented operators, i.e. ($I \otimes \sigma_1$), since we have a two-particle system.

$$|\phi_1\rangle = (I \otimes \sigma_1) \otimes |\phi_0\rangle = \frac{1}{\sqrt{2}}(|+\rangle|-\rangle + |-\rangle|+\rangle)$$

Think about that. The I in the operator preserves the state of the first particle. σ_1 flips up to down and vice versa, acting on the second particle. Similarly

$$|\phi_2\rangle = (I \otimes \sigma_2) \otimes |\phi_0\rangle = \frac{i}{\sqrt{2}}(|+\rangle|-\rangle - |-\rangle|+\rangle)$$

$$|\phi_3\rangle = (I \otimes \sigma_3) \otimes |\phi_0\rangle = \frac{1}{\sqrt{2}}(|+\rangle|+\rangle - |-\rangle|-\rangle)$$

Work backwards to the paired states. We'll need those for teleportation.

$$|+\rangle|+\rangle = \frac{1}{2}(|\phi_0\rangle + |\phi_3\rangle)$$

$$|+\rangle|-\rangle = \frac{1}{2}(|\phi_1\rangle - i|\phi_2\rangle)$$

$$|-\rangle|+\rangle = \frac{1}{2}(|\phi_1\rangle + i|\phi_2\rangle)$$

$$|-\rangle|-\rangle = \frac{1}{2}(|\phi_0\rangle - |\phi_3\rangle))$$

OK. Teleportation. Alice and Bob share a Bell state

$$|\phi_0\rangle = \frac{1}{\sqrt{2}}(|+\rangle|+\rangle + |-\rangle|-\rangle)$$

Alice grabs the state she wants to teleport to Bob.

$$|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$$

She interacts her states to form a tensor product $|\phi_0\rangle \otimes |\psi\rangle$. The subscripts below track spins held by Alice and Bob and, C, the spins to be teleported. The tensor product represents the whole system: spins of all three particles, the entangled pair and the state to be teleported.

$$|\phi_0\rangle_{AB} \otimes |\psi\rangle_C = \frac{1}{\sqrt{2}}\alpha(|+\rangle_A|+\rangle_C|+\rangle_B + |+\rangle_A|-\rangle_C|+\rangle_B)$$
$$+ \frac{1}{\sqrt{2}}\beta(|-\rangle_A|+\rangle_C|-\rangle_B + |-\rangle_A|-\rangle_C|-\rangle_B)$$

Well now. We can identify those leading $A \otimes C$ states with Bell bases.

$$|\phi_0\rangle_{AB} \otimes |\psi\rangle_C = \frac{1}{2}\alpha((|\phi_0\rangle + |\phi_3\rangle)_{AC}|+\rangle_B + (|\phi_1\rangle - i|\phi_2\rangle)_{AC}|+\rangle_B)$$
$$+ \frac{1}{2}\beta((|\phi_1\rangle + i|\phi_2\rangle)_{AC}|-\rangle_B + (|\phi_0\rangle - |\phi_3\rangle)_{AC}|-\rangle_B)$$

Regroup that last equation as factors of the Bell bases.

$$|\phi_0\rangle_{AB} \otimes |\psi\rangle_C = \frac{1}{2}|\phi_0\rangle_{AC}(\alpha|+\rangle_B + \beta|-\rangle_B) + \frac{1}{2}|\phi_1\rangle_{AC}(\alpha|-\rangle_B + \beta|+\rangle_B)$$
$$+ \frac{1}{2}i|\phi_2\rangle_{AC}(\alpha|-\rangle_B - \beta|+\rangle_B) + \frac{1}{2}|\phi_3\rangle_{AC}(\alpha|+\rangle_B - \beta|-\rangle_B)$$

Now look at that! Each term on the right is the AC Bell state times the associated spin operator (reflected in the signs; look close) on $|\psi\rangle_B$. Alice has teleported $|\psi\rangle$ to B! All Bob has to do is operate on his state with the appropriate spin operator. Alice has to send him that information, which operator. Done!

$$|\phi_0\rangle_{AB} \otimes |\psi\rangle_C = \frac{1}{2}\sum_{j=0}^{3}|\phi_j\rangle \otimes \sigma_j |\psi\rangle_B$$

Be sure to check Nielsen's quantum circuit for comparison. I think I finally see where that circuit comes from.

On to EPR and the Bell inequality. Zwiebach presents the argument really nicely. I've been confused about what is local realism and other such truck. Turns out it's right there in the two assumptions Einstein insists on:

1. Any measurement reflects a reality of the system. i.e., if your measurement determines that a particle has spin up, then that particle most definitely had spin up before the measurement. QM, of course, says that the particle was in a superposition of states before the measurement.
2. Conditions far away cannot affect measurements right here in the lab. QM, on the other hand, says particles can be entangled, i.e. correlated, over vast distances.

Bell's inequality established what's what. It's straightforward in Zwiebach's presentation. Key is that for a spin system the probability of measuring that spin lies along some axis at angle theta from the reference axis

$$P = \frac{1}{2} sin^2 \frac{\theta}{2}$$

Check the math. Maybe it's in Adams' notes a while back. But I'm pretty sure I verified this myself.

OK. Here's Professor Z's argument. Consider an experimental apparatus that can measure particle spin along any of three axes. Prepare entangled pairs. The table below lists all the possible entangled states. $+$ and $-$'s are spins along the three axes a, b, c. Columns list the measurement outcomes. State labels are arbitrary, just a counting device.

state	particle A	particle B
N_1	$+ + +$	$- - -$
N_2	$+ + -$	$- - +$
N_3	$+ - +$	$- + -$
N_4	$+ - -$	$- + +$
N_5	$- + +$	$+ - -$
N_6	$- + -$	$+ - +$
N_7	$- - +$	$+ + -$
N_8	$- - -$	$+ + +$

Figure the classical probabilities, what EPR predicted assuming local realism. Calculations list spin state for particle A followed by state for particle B.

$$a(+)b(+) = N_3 + N_4$$

$$a(+)c(+) = N_2 + N_4$$

$$b(+)c(+) = N_2 + N_6$$

From those equations, it's clear that

$$P[a(+)c(+)] \leq P[a(+)b(+)] + P[b(+)c(+)]$$

That's the classical prediction according to EPR. But QM says it ain't so! Suppose the angles between axes are small and the b axis lies between a and c .
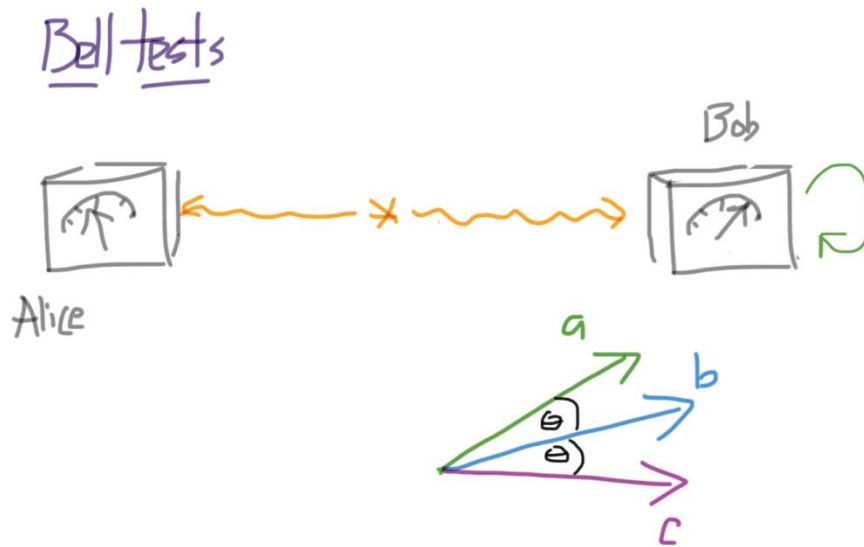

Figure. Experimental test of the Bell inequality. Alice and Bob independently measure spin orientation of their particle from an entangled pair. Bell inequality is obviously violated at small angle differences between the two measuring apparatus.

Those probabilities are

$$\frac{1}{2} sin^2\theta \ ? \leq \ \frac{1}{2} sin^2\frac{\theta}{2} + \frac{1}{2} sin^2\frac{\theta}{2} = sin^2\frac{\theta}{2}$$

Not so! At small angles

$$\frac{1}{2} sin^2\theta \cong \frac{1}{2}\theta^2 \geq \frac{1}{4}\theta^2$$

That's the QM prediction. Alain Aspect and many others have carried out the experiments. QM wins.

Lectures 21-23: Angular momentum

There's a whole lot of really dense definitions and proofs here. Main point is to develop angular momentum operators, general $J's$ that include all the maths of orbital angular momentum, spin, and everything.

Main point, after a lot of work (which I really need to figure out sometime) is

$$[J_i, J_j] = i\hbar\,\varepsilon_{ijk}\,J_k$$

Quantized. And note that $\hbar$ has units of angular momentum. Like $x \cdot p$ and $E \cdot t$. Think about those relations. Make them all operators. $\hat{x} \cdot \hat{p}$ is obvious; that's the classical angular momentum. $\hat{E} \cdot \hat{t}$ needs some more thinking.

Anyway, after all the maths gymnastics we end up with a Hamiltonian for a spin in a central potential. Like an electron in the electromagnetic potential of its nucleus.

$$H = \frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{\hbar^2}{2mr^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}\right) + V(r)$$

Note all those accelerations along the radial and angular directions. That seems a quick shorthand to think about it, anyway.

From that Hamiltonian you can show that the energy is quantized with quantum numbers j in increments of $\frac{1}{2}\hbar$ components along the z-axis and total momentum between $\mp j$. I think that about captures it.

The generalized wavefunction, with all the angular and radial terms collected into Y and u operators (via algebra to combine all the messy coefficients)

$$\psi_{Elm} = \frac{u_{El}(r)}{\hat{r}} + Y_{lm}(\theta,\rho)$$

ρ is a function of radial distance. The wavefunction depends on radial distance and spherical angle. There's just a whole lot of calculation goes into figuring those. From that we get the Hamiltonian

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}u_E(r) + V_{eff}u_{El} = Eu_{El}$$

where the effective potential includes a term for centrifugal force

$$V_{eff} = V(r) - \frac{\hbar^2 l(l+1)}{2mr^2}$$

Take-home from all this – Zwiebach's words – is summarized in the graph for orbital angular momentum. Note we're talking orbital L here, not spin, so energy levels are unit quantized and not half-integer. Beyond the l_{i0} (i^{th} energy level with zero angular momentum, as in s^1, s^2, s^3, etc. orbitals) the states are degenerate. So, for example, there are three states at each l_{i1}, one for each of the three l_z and with components labeled by m in the algebra. (Note that there are $m = 2l + 1$ z-components of angular momentum. See why?)
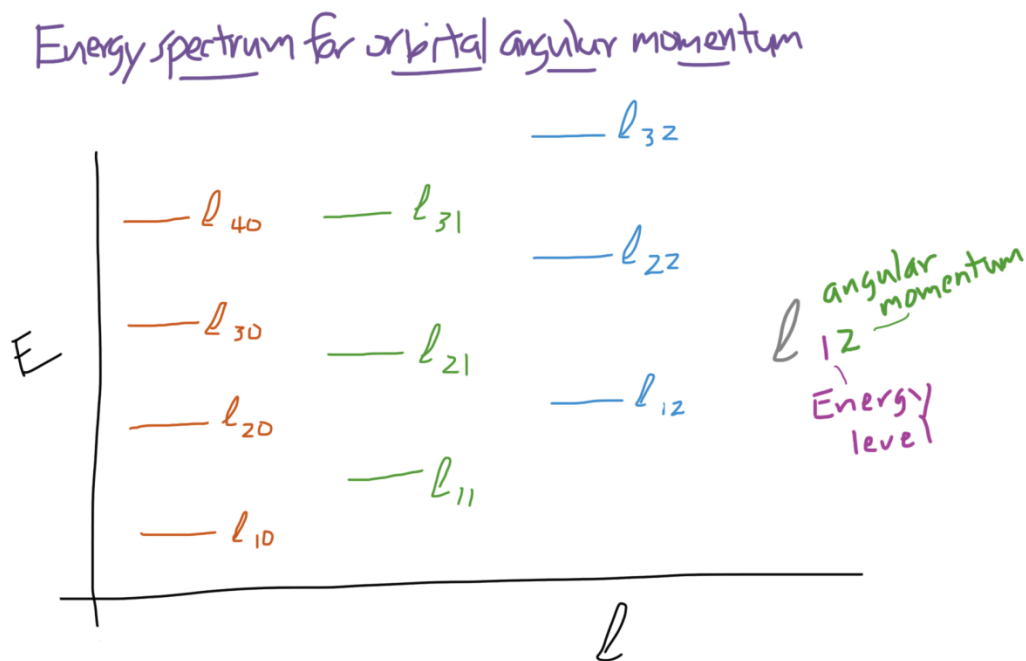
Take a look here how the spectrum is higgledy-piggledy, not nice and neat like the QHO spectrum below or the Hydrogen spectrum.

Interesting also is that calculating the wavefunctions on a 2-d surface requires 3-d angular momentum. In fact, the L_{xyz} operators emerge naturally from the algebra. (Don't ask me to demonstrate that right off. Check out the lecture notes.)

What do those wavefunctions look like? 3-d angular momentum? Well, they're Bessel functions. Those are the 3-d standing waves, e.g. representing the modes of vibration of our Mr. Sun. Below is a vibrating membrane, but you get the idea.
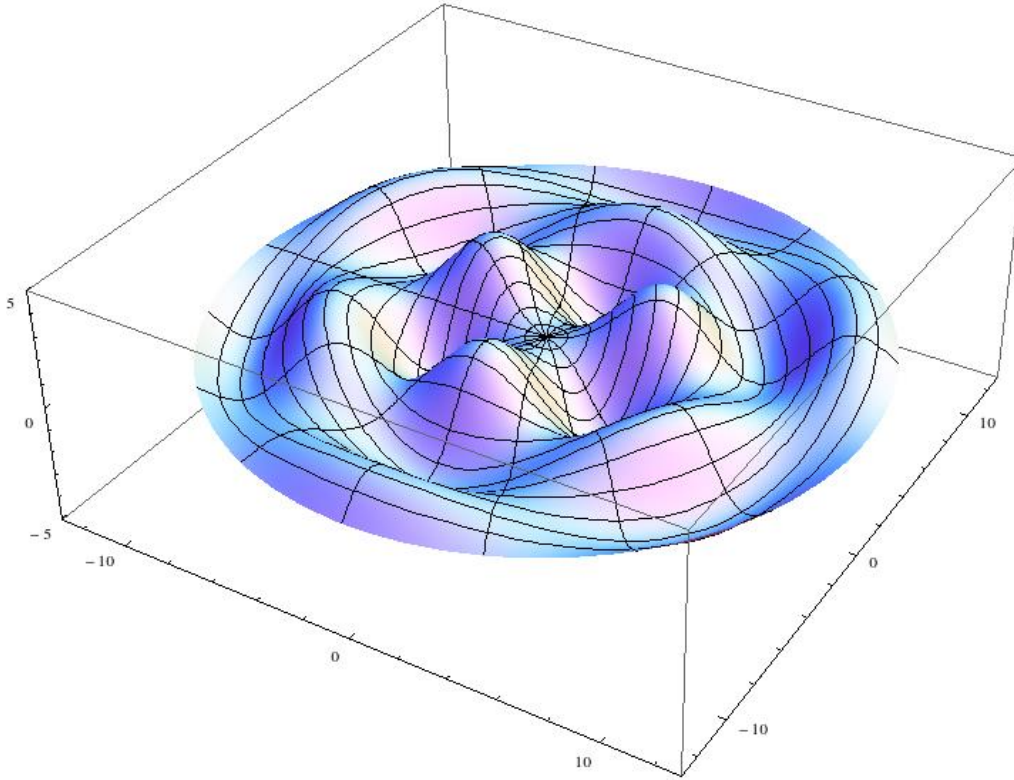
Figure: Vibrating drum Bessel function. Credit Slawomir Bialy. Bessel Function. Wikipedia.

Turns out wavefunctions in a uniform spherical potential well are a mess, no pattern. But the spectrum of a 3d quantum harmonic oscillator is nice and tidy. Here's the algebra. Hamiltonian has the same form expressed in 3d. Then it's all numerology.

$$H = \frac{\hat{p}^2}{2m} - \frac{1}{2}m\omega^2\hat{r}^2$$

We build the spectrum of the 3d QHO like we did the 1d oscillator but with two more sets of ladder operators. And, note this, those operators form entangled states. The states in the QHO are tensor products. That's interesting; before we were entangling particles. Now we're entangling operators.

Start at the ground state, no angular momentum. $|\psi\rangle = |0\rangle$. In 3d that has energy $\frac{3}{2}\hbar\omega$. Now spin it up to one unit of angular momentum. There are three possible spin states, $a_x^\dagger|0\rangle$, $a_y^\dagger|0\rangle$, and $a_z^\dagger|0\rangle$ all with l_1 . Degenerate in energy $\frac{5}{2}\hbar\omega$.

l_2 gets trickier. Six possible states, combinations of the creation operators adding to two units of half-spin.

$$a_x^\dagger a_x^\dagger|0\rangle, a_x^\dagger a_y^\dagger|0\rangle, a_y^\dagger a_y^\dagger|0\rangle, a_y^\dagger a_z^\dagger|0\rangle, a_z^\dagger a_z^\dagger|0\rangle, a_x^\dagger a_z^\dagger|0\rangle$$

Well now. There's a problem. Six states totals energy $\frac{6}{2}\hbar\omega$. Ain't no such energy on the spectrum. We've got to split that degeneracy. Solution is five l_2 states and one l_0 all at $E = \frac{7}{2}\hbar\omega$. Onward and you get a spectrum that looks like

Spectrum of 3d SHO
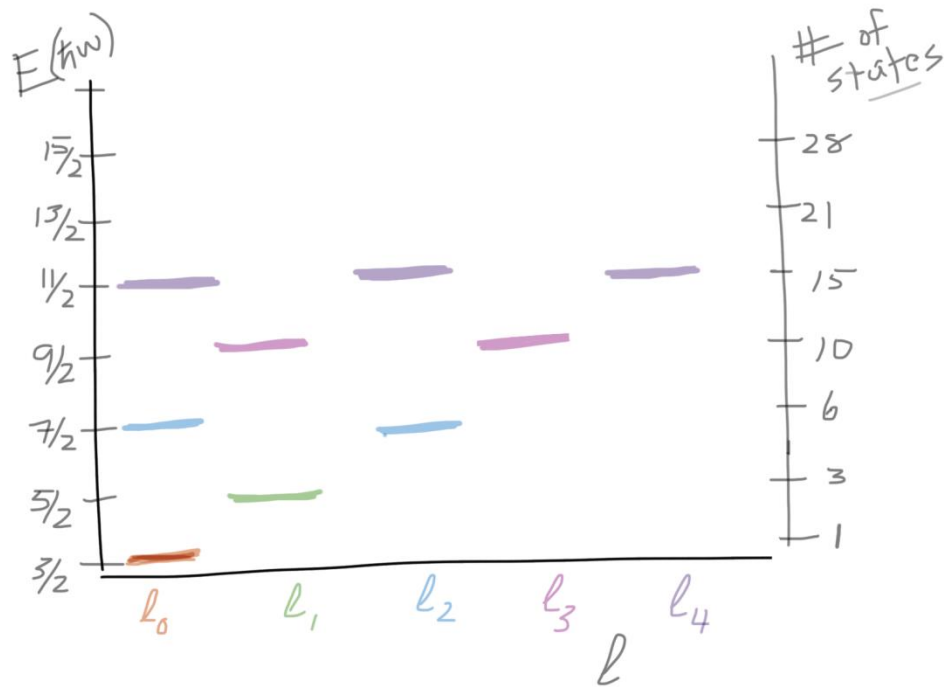
But hold on here. We built those six states in l_2. How did we end up with a state in l_0? The answer (I think – Prof. Z. didn't address this directly) is entanglement. Entangle all those l_2 states and you get an isotropic system among them. It's spherically symmetric, no preferred direction.

On then to the hydrogen spectrum. With all these tools available it's simple! I'd assumed H would be a colossal maths challenge. Not so! It all spills out of the Hamiltonian of a central electric potential.

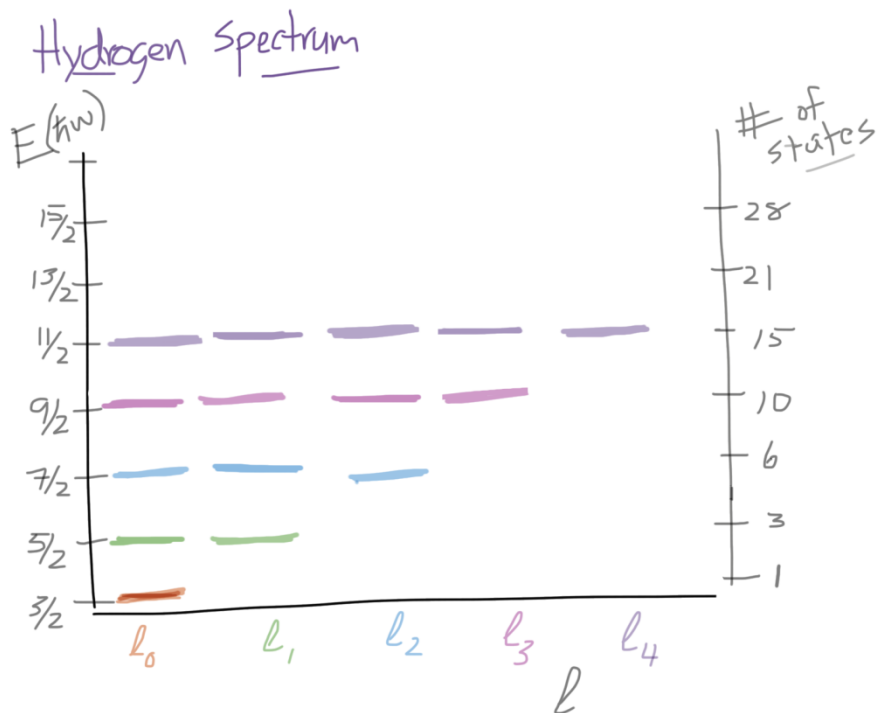$$H = \frac{\hat{p}^2}{2m} - \frac{e^2}{\hat{r}} = \frac{\hat{\hbar}^2}{2m\hat{r}^2} - \frac{e^2}{\hat{r}}$$

We can solve for the Bohr radius immediately. Set the potential and kinetic energies equal. Solve for radial distance.

$$a_0 = \frac{\widehat{\hbar}^2}{2me^2}$$

Use that to solve the ground state energy: plug a_0 into the potential, then ladder up the energy spectrum with the angular momentum operators! Same drill.

$$E_{nl} = -\frac{e^2}{2a_0}\frac{1}{n^2}$$

Surprising is how neat and tidy is the hydrogen spectrum. All kinds of degeneracy. You can see all the orbitals right there in the spectrum.

Hydrogen Spectrum

$E(\hbar\omega)$ # of states

E		# states
$17/2$		28
$13/2$		21
$11/2$		15
$9/2$		10
$7/2$		6
		3
$5/2$		
$3/2$		1

$\ell_0 \quad \ell_1 \quad \ell_2 \quad \ell_3 \quad \ell_4$

ℓ

Lecture 24: Intro to perturbation theory

Here's interesting: the Feynman-Hellman theory. RPF figured it out as an undergrad.

Idea is that if you know the state of a system and you tickle it, say, with an extra potential you can add the perturbation to the operators on the initial state and get a close approximation to the perturbed state. Zwiebach's example here is fine splitting in the hydrogen spectrum due to the magnetic moment of the electron.

Feynman-Hellman says

$$\frac{\partial E(\lambda)}{\partial \lambda} = \left\langle \psi(\lambda) \left| \frac{\partial \widehat{H}(\lambda)}{\partial \lambda} \right| \psi(\lambda) \right\rangle$$

where λ is the perturbation of the magnetic moment. Simple enough, and it makes sense. You can check it out with the bra and ket algebra.

Now suppose $\widehat{H}_{new} = \widehat{H}_0 + \frac{\partial \widehat{H}(\lambda)}{\partial \lambda} d\lambda = \widehat{H}_0 + \lambda \widehat{H}_0$. In the case of fine splitting, all we have to calculate is that second term. It's the change in energy due to interaction of spin orbital momentum with the magnetic moment of the electron. That's what we'll add as a perturbation.

$$V(L \cdot S) = -\frac{\hbar^2}{2mc^2} \widehat{L} \cdot \widehat{S}$$

where the orbital L and spin operators can be calculated from our previous calculations of the orbital radius and the fine structure constant. Not bad!

Lecture 24: Spin-orbit coupling

Question is: what happens when the system includes multiple components. For example, the hydrogen atom has a central potential, the electromagnetic field anchored on the proton, plus the magnetic moment of the electron in that field. The motion of the electron induces a magnetic field, and μ_e interacts with that field. That's the perturbation conditions introduced above. Now take a look at the states and their energies.

Represent the possible $|l, m\rangle$ orbital angular momentum states. l is total angular momentum; m is the z component. There are three possible states for $l = 1$.

$$|1, 1\rangle, |1, 0\rangle, \text{and } |1, -1\rangle$$

And there are two possible electron spin states.

$$\left| \frac{1}{2}, \frac{1}{2} \right\rangle \text{ and } \left| \frac{1}{2}, -\frac{1}{2} \right\rangle$$

Altogether, then, there are six possible spin-orbital terms in the Hamiltonian.

$$|1, 1\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle$$

$$|1, 1\rangle \otimes \left| \frac{1}{2}, -\frac{1}{2} \right\rangle$$

$$|1, 0\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle$$

$$|1,0\rangle \otimes \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$$

$$|1,-1\rangle \otimes \left|\frac{1}{2}, \frac{1}{2}\right\rangle$$

$$|1,-1\rangle \otimes \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$$

The top and bottom states have energy $\frac{3}{2}\hbar\omega$. The middle four form a degenerate multiplet with energy $\frac{1}{2}\hbar\omega$.

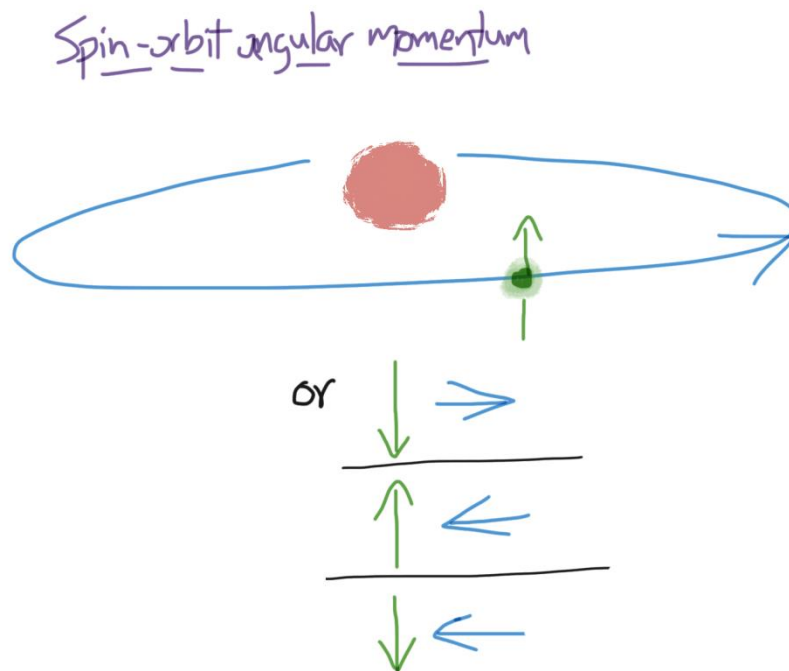It's all wavefunctions, but we can see the picture in a cartoon.

Figure. Spin-orbit angular momentum. Depending on magnetic moments of the nucleus and electron, relative orientation of spin and orbital angular momentum split the spectrum into fine and hyperfine spectra.

Lecture 26: The hydrogen spectrum

Done! Last of the lectures! And what a great lecture series! Prof. Zwiebach and his assistants do a marvelous job presenting the quantum mechanics.

Can I do the calculations? No, not well. I need to really sit down and practice. Do the problem sets. Do the exams.

But I think I have a whole lot better understanding of the concepts. I learned tons about vector spaces and operators. I learned tons about wavefunctions and about Hamiltonians and complete sets of commuting operators. Tons.

The hydrogen spectrum requires a bunch of vector algebra. Blackboard after blackboard of equations. What it all comes down to, I think is:

1. Look at those 3d Bessel functions. In them you can see the angular momentum, total and z-component.
2. Include the additional angular momentum from the Runge-Lenz vector.
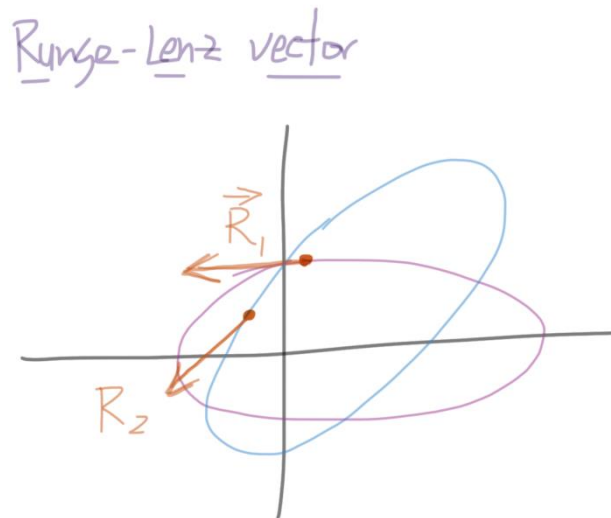
Runge-Lenz $\vec{R}$ points along the major axis of an ellipse. It has constant magnitude, depending on $\hat{p}$. If the orbital is precessing, as in the presence of the orbital B field, then that precession, captured by Runge-Lenz, contributes to total angular momentum.

Suppose the electron has an orbital angular momentum $\hat{L}$ and spin $\hat{S}$ with total $\hat{J} = \hat{L} + \hat{S}$. Now suppose the whole system is precessing, major axis of the atom revolving around the nucleus. We have to include those effects, the precession, in calculations of angular momentum. That takes the form $\hat{R} \times \hat{J}$. There's a whole bunch of commutation relations in there, and it all shows up in the spectrum.

Meantime I'm off to do taxes and get ready for a Grand Canyon trip. Hasta luego!

Epilogue

Grand Canyon has been postponed because of CoVid-19. School and community life, all have been postponed.

But there's still quantum mechanics to learn. I'm embarking on Scott Aaronson's lectures for quantum computing. Hasta hasta luego!